A NEW GENERALIZATION OF THE NARAYANA NUMBERS INSPIRED BY LINEAR OPERATORS ON ASSOCIATIVE d-ARY ALGEBRAS

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Abstract. We introduce and study a generalization of the Narayana numbers $N_d(n,k) = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+(n-k)(d-2)+1}{k}$ for integers $d \geq 2$ and $n, k \geq 0$. This two-parameter array extends the classical Narayana numbers ($d = 2$) and yields a d-ary analogue of the Catalan numbers $C_d(n) = \sum_{k=0}^{n} N_d(n,k)$. We give nine combinatorial interpretations of $N_d(n,k)$ that unify and generalize known combinatorial interpretations of the Narayana numbers and $C_3(n)$ in the literature. In particular, we show that $N_d(n,k)$ counts a natural class of operator monomials over a d-ary associative algebra, thereby extending a result of [10] for the binary case. We also construct explicit bijections between these monomials and several families of classic combinatorial objects, including Schröder paths, Dyck paths, rooted ordered trees, and 231-avoiding permutations.

Contents

1. Introduction

Given an integer $d \geq 2$ and integers $n, k \geq 0$, define

$$N_d(n,k) := \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+(n-k)(d-2)+1}{k}.$$

Table 1 lists $N_2(n,k)$ and $N_3(n,k)$ for several small values of n and k.

When $d = 2$, we obtain the well-studied Narayana numbers (sequence A001263 from the Online Encyclopedia of Integer Sequences (OEIS) [26]), named after the 20th-century Indo-Canadian mathematician Tadepalli Venkata Narayana. For an introduction to the Narayana numbers, see, for instance, [27, Chapter 2] or [19, Chapter 33]. The sequences $N_d(n,k)$ for $d \geq 3$ were not in the

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$n \setminus k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	3	1					
3	1	6	6	1				
4	1	10	20	10	1			
5	1	15	50	50	15	1		
6	1	21	105	175	105	21	1	
7	1	28	196	490	490	196	28	1

$n \setminus k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	4	1					
3	1	9	10	1				
4	1	16	42	20	1			
5	1	25	120	140	35	1		
6	1	36	275	600	378	56	1	
7	1	49	546	1925	2310	882	84	1

TABLE 1. The numbers $N_2(n,k)$ (left) and $N_3(n,k)$ (right) for $0 \le k \le n \le 7$

OEIS at the start of this research project; however, $N_3(n,k)$ was briefly mentioned in Callan and Mansour [14, Section 3.16] and studied in more depth in Huh et al. [21]. In preparation for this manuscript, we have submitted the sequences $N_3(n,k)$ (A391045), $N_4(n,k)$ (A391046), $N_5(n,k)$ (A391047), and $N_6(n,k)$ (A391048) to the OEIS. (The sequence $N_3(n,k)$ has been applied by the second author [8] in work on algebraic identities for linear operators.) For several other generalizations of the Narayana numbers, see [4, 13, 22, 29]. Additionally, for the integer sequence named after the 14th-century Indian mathematician Narayana Pandita, which is sometimes also referred to as the Narayana numbers in the literature, see [2] and A000930 from the OEIS.

It is well-known that each row of Narayana numbers sums to the familiar Catalan numbers (A000108). Consequently, our generalized Narayana numbers naturally lead to a d-ary generalization of the Catalan numbers. More precisely, for $d \ge 2$, we define

$$C_d(n) := \sum_{k=0}^{n} N_d(n,k).$$

Table 2 lists the first few entries of $C_d(n)$ for $d \le 6$. Notably, $C_2(n)$ corresponds to the Catalan numbers, which admits over 200 combinatorial interpretations [28]. The sequence $C_3(n)$ has also been studied in various combinatorial contexts [1, 3, 5, 6, 7, 11, 12, 14, 15, 17, 20, 21, 24, 25, 31] — see also A106228 for additional properties of this sequence. The sequences for $4 \le d \le 6$ appear in the OEIS but lack substantial combinatorial interpretation. As a consequence of our work on $N_d(n,k)$, this paper connects those sequences to a variety of combinatorial objects.

$d \setminus n$	0	1	2	3	4	5	6	7	OEIS
2	1	2	5	14	42	132	429	1430	A000108
3	1	2	6	21	80	322	1347	5798	A106228
4	1	2	7	29	131	627	3124	16032	A300048
5	1	2	8	38	196	1073	6120	35968	A364723
6	1	2	9	48	276	1687	10750	70597	A364734

TABLE 2. The generalized Catalan numbers $C_d(n)$

In this manuscript, we study the generalized Narayana numbers $N_d(n,k)$ and present nine combinatorial interpretations of these numbers. First, in Section 2, we demonstrate that $N_d(n,k)$ counts a specific set of operator monomials over a d-ary algebra, generalizing a result by the second author and Elgendy [10], who established this for the case when $d = 2$.

In Section 3, we provide bijections from these d-ary operator monomials to several classic combinatorial objects, showing that $N_d(n,k)$ also counts certain subsets of Schröder paths, rooted ordered trees, Dyck paths, and 231-avoiding permutations.

In Section 4, we provide four additional combinatorial interpretations of $N_d(n,k)$ that generalize known interpretations for $N_3(n,k)$ studied in Huh et al. [21]. These interpretations involve Schröder paths with labelled descents, a family of lattice paths generalizing the F-paths defined in Huh et al., labelled Dyck paths, and labelled ordered trees. Our new interpretations also specialize to classic interpretations for Narayana and Catalan numbers when $d = 2$. We conclude with some potential future research directions in Section 5.

By studying $N_d(n,k)$, we aim to generalize some classic results on Narayana and Catalan numbers. Moreover, under this framework, the well-studied sequence A106228 can now be seen as a ternary extension of the classic Catalan numbers, offering new perspectives on this and related sequences. The mapping we provide between d-ary operator monomials and other well-studied combinatorial objects may also be beneficial in exploring related problems in algebraic operads. In particular, since the d-ary operator monomials form a basis for the nonsymmetric operad generated by one unary operation and one associative d-ary operation, the bijections we construct in this paper between these operator monomials and other combinatorial objects will aid in the understanding of the d-ary analogues of well-known binary operads.

2. d-ary operator monomials

In this section, we provide the first combinatorial interpretation of $N_d(n,k)$; namely, the interpretation in terms of d-ary operator monomials.

Let $\mathcal{A}$ be an associative d-ary algebra. Specifically, $\mathcal{A}$ is a vector space equipped with a multilinear map $B\colon \mathcal{A}^d \to \mathcal{A}$ with product denoted $B(a_1,\ldots,a_d) \longmapsto a_1 \cdots a_d$ that satisfies d-ary associativity. That is, for every $a_1,\ldots,a_{2d-1} \in \mathcal{A}$ and indices $0 \le i < j \le d-1$,

$$a_1 \cdots a_i(a_{i+1} \cdots a_{i+d})a_{i+d+1} \cdots a_{2d-1} = a_1 \cdots a_j(a_{j+1} \cdots a_{j+d})a_{j+d+1} \cdots a_{2d-1}.$$

It is easy to see that any monomial in a d-ary operation must have $k(d-1)+1$ indeterminates for some $k \ge 0$; that is, the number of indeterminates must be congruent to 1 modulo $d-1$. (Thus for $d = 2$ this is no restriction, and for $d = 3$ it says that the number of indeterminates must be odd.) The d-ary associativity identities imply by the usual inductive argument on k that in any d-ary monomial we may unambiguously remove the parentheses.

Associative d-ary algebras were first studied by Carlsson [16]; for a more modern operadic point of view see Gnedbaye [18]. For background on operads, see Loday and Vallette [23] for the theoretical aspects, and Bremner and Dotsenko [9] for the algorithmic aspects.

Next, let $L\colon \mathcal{A} \to \mathcal{A}$ be a linear operator (unary operation) on the underlying vector space of the associative d-ary algebra $\mathcal{A}$. We define the set of *(d-ary) operator monomials* $\mathcal{M}_d$ recursively as follows:

- a single indeterminate (regarded as a generic element of $\mathcal{A}$) is an operator monomial;
- if $M_1,\ldots,M_d$ are operator monomials, then $M_1 \cdots M_d$ (using the d-ary associative product in $\mathcal{A}$) is also an operator monomial;
- if M is an operator monomial, then $L(M)$ is also an operator monomial.

Given an operator monomial $M \in \mathcal{M}_d$, we let $\mathrm{topt}(M)$ (for *total operations*) denote the total number of operations which appear (counting both B and L), and $\mathrm{lopt}(M)$ (for *linear operations*) denote the number of occurrences of the linear operator L. For example, Table 3 lists the ternary ($d = 3$) operator monomials M with $\mathrm{topt}(M) = 3$, where we use the standard abbreviations such as $L(L(a)) = L^2(a)$.

operator monomials M	$\operatorname{lopt}(M)$	count
$a_1a_2a_3a_4a_5a_6a_7$	0	1
$L(a_1a_2a_3a_4a_5)$, $L(a_1a_2a_3)a_4a_5$, $L(a_1)a_2a_3a_4a_5$, $a_1L(a_2a_3a_4)a_5$, $a_1L(a_2)a_3a_4a_5$, $a_1a_2L(a_3a_4a_5)$, $a_1a_2L(a_3)a_4a_5$, $a_1a_2a_3L(a_4)a_5$, $a_1a_2a_3a_4L(a_5)$	1	9
$L^2(a_1a_2a_3)$, $L(L(a_1)a_2a_3)$, $L^2(a_1)a_2a_3$, $L(a_1L(a_2)a_3)$, $L(a_1a_2L(a_3))$, $L(a_1)L(a_2)a_3$, $L(a_1)a_2L(a_3)$, $a_1L^2(a_2)a_3$, $a_1L(a_2)L(a_3)$, $a_1a_2L^2(a_3)$	2	10
$L^3(a_1)$	3	1

TABLE 3. The 21 ternary operator monomials $M \in \mathcal{M}_3$ with $\operatorname{topt}(M) = 3$

Observe that for every $M \in \mathcal{M}_d$, we have $0 \leq \operatorname{lopt}(M) \leq \operatorname{topt}(M)$. Next, we let $\deg(M)$ denote the *degree* of M, which is the number of indeterminates involved in M. It follows that

$$\deg(M) = (\operatorname{topt}(M) - \operatorname{lopt}(M))(d-1) + 1 \tag{1}$$

for every $M \in \mathcal{M}_d$. To understand this, note that M contains exactly $\operatorname{topt}(M) - \operatorname{lopt}(M)$ instances of the associative d-ary operation, and each application of that operation "multiplies down" d inputs into one output. An immediate consequence of (1) is that $\deg(M)$ is congruent to 1 modulo $d-1$ for every $M \in \mathcal{M}_d$. In fact, to produce all operator monomials of degree $k(d-1)+1$ (for any given integer $k \geq 1$), we can start with the generic monomial $a_1 \cdots a_{k(d-1)+1}$ and insert operator symbols L in all possible ways, respecting the fact that any product must contain a number of factors congruent to 1 modulo $d-1$.

Next, we show that the set of monomials in $\mathcal{M}_d$ with a fixed number of operations is counted by $N_d(n,k)$.

Theorem 1. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then $N_d(n,k)$ counts the number of d-ary operator monomials $M \in \mathcal{M}_d$ such that* $\operatorname{topt}(M) = n$ *and* $\operatorname{lopt}(M) = k$.

Theorem 1 extends a result by the second author and Elgendy [10, Lemma 2.5], who proved the result for the case $d = 2$. Before we prove Theorem 1, we first derive a functional equation for the generating function of $N_d(n,k)$. Let

$$A_d(x,y) \coloneqq \sum_{n \geq 0} \sum_{k=0}^{n} N_d(n,k) x^n y^k.$$

For example, for $d = 2$, we have

$$A_2(x,y) = 1 + (1+y)x + (1+3y+y^2)x^2 + (1+6y+6y^2+y^3)x^3 + \cdots$$

We then have the following.

Proposition 2. *For every integer $d \geq 2$,*

$$A_d(x,y) = (1 + xyA_d(x,y)) \left(1 + xA_d(x,y)\,(1 + xyA_d(x,y))^{d-2}\right). \tag{2}$$

Proof. Let u be a function of x and y that satisfies the functional equation

$$u = (1 + xyu)\left(1 + xu(1 + xyu)^{d-2}\right). \tag{3}$$

Next, let $v \coloneqq xyu$. Multiplying both sides of (3) by xy and simplifying, we obtain

$$v = x(1 + v)(y + v(1 + v)^{d-2}). \tag{4}$$

To prove our claim, it suffices to show that $[x^{n+1}][y^{k+1}]v = N_d(n, k)$ for every $n, k \geq 0$ and $d \geq 2$. Observe that (4) can be rewritten as $v = x\Phi(v)$, where $\Phi(t) \coloneqq (1 + t)\left(y + t(1 + t)^{d-2}\right)$. Now, $\Phi(0) = y$, which we may assume to be nonzero. We will now proceed to obtain the coefficients of v via Lagrange inversion (see, for instance, [30, Chapter 5] for a reference).

$$\begin{aligned}
[x^{n+1}][y^{k+1}]v &= [y^{k+1}][t^n]\frac{1}{n+1}\Phi(t)^{n+1} \\
&= [y^{k+1}][t^n]\frac{1}{n+1}(1+t)^{n+1}\left(y + t(1+t)^{d-2}\right)^{n+1} \\
&= [y^{k+1}][t^n]\frac{1}{n+1}(1+t)^{n+1}\left(\sum_{i=0}^{n+1}\binom{n+1}{i}y^i t^{n+1-i}(1+t)^{(n+1-i)(d-2)}\right) \\
&= [y^{k+1}][t^n]\frac{1}{n+1}\left(\sum_{i=0}^{n+1}\binom{n+1}{i}y^i t^{n+1-i}(1+t)^{n+(n+1-i)(d-2)+1}\right) \\
&= [t^n]\frac{1}{n+1}\binom{n+1}{k+1}t^{n-k}(1+t)^{n+(n-k)(d-2)+1} \\
&= [t^k]\frac{1}{n+1}\binom{n+1}{k+1}(1+t)^{n+(n-k)(d-2)+1} \\
&= \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+(n-k)(d-2)+1}{k} \\
&= N_d(n,k).
\end{aligned}$$

This concludes the proof. □

Next, given $M \in \mathcal{M}_d$, we say that M is *irreducible* if it satisfies one of the following conditions:

- It is a single indeterminate (in which case $\operatorname{topt}(M) = \operatorname{lopt}(M) = 0$), or
- $M = L(M')$ for some operator monomial $M' \in \mathcal{M}_d$.

We let $\overline{\mathcal{M}}_d \subseteq \mathcal{M}_d$ denote the set of irreducible d-ary operator monomials. Additionally, given a positive integer n, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$. We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $d \geq 2$ be fixed, and consider the generating function

$$u \coloneqq \sum_{M \in \mathcal{M}_d} x^{\operatorname{topt}(M)} y^{\operatorname{lopt}(M)}.$$

To prove our claim, we will show that u satisfies the same functional equation as $A_d(x, y)$ from Proposition 2, which would imply that $[x^n][y^k]u = N_d(n, k)$ for every integer $n, k \geq 0$. To derive a functional equation for u, we first observe that

$$\sum_{M \in \overline{\mathcal{M}}_d} x^{\operatorname{topt}(M)} y^{\operatorname{lopt}(M)} = 1 + xyu. \tag{5}$$

To see this, note that if $M \in \overline{\mathcal{M}}_d$ is a single indeterminate, then M contributes

$$x^{\mathrm{topt}(M)} y^{\mathrm{lopt}(M)} = x^0 y^0 = 1$$

to the sum. Otherwise, $M \in \overline{\mathcal{M}}_d' \coloneqq \{L(M') : M' \in \mathcal{M}_d\}$, and we have

$$\sum_{M \in \overline{\mathcal{M}}_d'} x^{\mathrm{topt}(M)} y^{\mathrm{lopt}(M)} = \sum_{M' \in \mathcal{M}_d} x^{\mathrm{topt}(M')+1} y^{\mathrm{lopt}(M')+1} = xyu.$$

Now, given an integer $\ell \geq 1$, let $\mathcal{M}_{d,\ell} \subseteq \mathcal{M}_d$ be the set of operator monomials that can be expressed as the product of ℓ irreducible operator monomials. Notice that every $M \in \mathcal{M}_d$ is contained in $\mathcal{M}_{d,j(d-1)+1}$ for a unique integer $j \geq 0$. Thus, $\mathcal{M}_d$ is equal to the disjoint union $\bigcup_{j \geq 0} \mathcal{M}_{d,j(d-1)+1}$.

Now, given $M \in \mathcal{M}_d$, if we write $M = M_1 M_2 \cdots M_{j(d-1)+1}$ where M_i is irreducible for every $i \in [j(d-1)+1]$, then

$$\mathrm{topt}(M) = j + \sum_{i=1}^{j(d-1)+1} \mathrm{topt}(M_i), \quad \mathrm{lopt}(M) = \sum_{i=1}^{j(d-1)+1} \mathrm{lopt}(M_i). \tag{6}$$

Combining (5) and (6), we see that, for every $j \geq 0$, we have

$$\sum_{M \in \mathcal{M}_{d,j(d-1)+1}} x^{\mathrm{topt}(M)} y^{\mathrm{lopt}(M)} = x^j \left(1 + xyu\right)^{j(d-1)+1}.$$

Putting everything together, we obtain that

$$\begin{aligned} u &= \sum_{M \in \mathcal{M}_d} x^{\mathrm{topt}(M)} y^{\mathrm{lopt}(M)} \\ &= \sum_{j \geq 0} \sum_{M \in \mathcal{M}_{d,j(d-1)+1}} x^{\mathrm{topt}(M)} y^{\mathrm{lopt}(M)} \\ &= \sum_{j \geq 0} x^j \left(1 + xyu\right)^{j(d-1)+1} \\ &= \frac{1 + xyu}{1 - x(1 + xyu)^{d-1}}. \end{aligned}$$

Next, observe that

$$\begin{aligned} u &= \frac{1 + xyu}{1 - x(1 + xyu)^{d-1}}, \\ u - xu(1 + xyu)^{d-1} &= 1 + xyu, \\ u &= 1 + xyu + xu(1 + xyu)^{d-1}, \\ u &= (1 + xyu)\left(1 + xu(1 + xyu)^{d-2}\right). \end{aligned}$$

Observe that u indeed satisfies the same functional equation (2) as $A_d(x,y)$. Thus, we obtain that $[x^n][y^k]u = N_d(n,k)$, and our claim follows. $\square$

It is obvious that if we set $y = 1$ in $A_d(x,y)$, we obtain the generating function for our generalized Catalan numbers $C_d(n)$. Thus, Proposition 2 and the proof of Theorem 1 readily imply the following.

Corollary 3. *Let $d \geq 2$ be an integer, and define the generating function $u := \sum_{n \geq 0} C_d(n)x^n$. Then u satisfies the functional equation*

$$u = (1 + xu)\left(1 + xu(1 + xu)^{d-2}\right),$$

or equivalently,

$$u = \frac{1 + xu}{1 - x(1 + xu)^{d-1}}.$$

We next highlight a simple identity involving $N_d(n, k)$.

Proposition 4. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then*

$$N_2(n, k) \leq N_d(n, k) \leq N_2((n - k)(d - 1) + k, k).$$

Proof. While it is relatively straightforward to derive both inequalities algebraically from the definition of $N_d(n, k)$, we present a combinatorial argument. We first prove $N_2(n, k) \leq N_d(n, k)$ by establishing an injection from $\mathcal{M}_2$ to $\mathcal{M}_d$. Given $M \in \mathcal{M}_2$, let $a_1, \ldots, a_\ell$ be the indeterminates involved in M. Let M' be the operator monomial obtained from M by replacing every instance of a_i with a product of d indeterminates $a_{i1}a_{i2}\cdots a_{id}$ for every $i \in \{2, \ldots, \ell\}$. Then $M' \in \mathcal{M}_d$ with $\operatorname{topt}(M') = \operatorname{topt}(M)$ and $\operatorname{lopt}(M') = \operatorname{lopt}(M)$, and the inequality follows.

Next, we prove $N_d(n, k) \leq N_2((n - k)(d - 1) + k, k)$ with an injection from $\mathcal{M}_d$ to $\mathcal{M}_2$. Let $M \in \mathcal{M}_d$ where $\operatorname{topt}(M) = n$ and $\operatorname{lopt}(M) = k$. Then $\deg(M) = 1 + (n - k)(d - 1)$. Now let $M' \in \mathcal{M}_2$ be the same expression as M, but parsed using a binary operation instead of a d-ary operation. Then $\operatorname{lopt}(M') = \operatorname{lopt}(M) = k$, and $\operatorname{topt}(M') = (n - k)(d - 1) + k$. This completes the proof. $\square$

Proposition 4 offers two avenues for finding combinatorial interpretations of $N_d(n, k)$. First, we can start with a set of combinatorial objects with cardinality $N_2((n - k)(d - 1) + k, k)$, and then *restrict* it to a particular subset of size $N_d(n, k)$. All four combinatorial interpretations we provide for $N_d(n, k)$ in Section 3 belong to this category.

Alternatively, one could begin with a set of size $N_2(n, k)$ and then *replicate* its elements (e.g., by introducing labels to differentiate between them) to create an expanded set of size $N_d(n, k)$. Three of the four combinatorial interpretations discussed in Section 4 (see Theorems 10, 11, and 13 in particular) can be viewed from this perspective.

Since the next two sections will involve several sets of combinatorial objects and mappings between them, we have organized how the bijections $f_1, \ldots, f_8$ relate to these combinatorial objects in Figure 1 for ease of reference. These objects and the bijections will be fully defined in Sections 3 and 4.

$$\begin{tikzcd}
\mathcal{S}_d & & \mathcal{T}_d \arrow[r, "f_3"] & \mathcal{Q}_d \arrow[r, "f_4"] & \mathcal{P}_d \\
& \mathcal{M}_d \arrow[ul, "f_1"'] \arrow[ur, "f_2"] \arrow[dl, "f_5"] \arrow[dr, "f_6"'] & & & \\
\widetilde{\mathcal{S}}_d & & \mathcal{F}_d \arrow[r, "f_7"] & \widetilde{\mathcal{Q}}_d \arrow[r, "f_8"] & \widetilde{\mathcal{T}}_d
\end{tikzcd}$$

FIGURE 1. Relating combinatorial objects and associated bijections

3. Several "restricted" combinatorial interpretations of $N_d(n,k)$

In this section, we provide four sets of combinatorial objects that are counted by $N_d(n,k)$ by establishing bijections among these sets and the d-ary operator monomials $\mathcal{M}_d$.

3.1. **Schröder paths with restricted horizontal steps.** A *Schröder path* is a lattice path that begins at $(0,0)$, consists only of up steps $U=(1,1)$, down steps $D=(1,-1)$, and horizontal steps $H=(2,0)$, remains on or above the x-axis, and ends on the x-axis. Observe that a Schröder path P must end at $(2n,0)$ for some non-negative integer n, which is often referred to as the *semilength* of P and denoted $|P|$. Additionally, given an up step U in a Schröder path that goes from $y=\ell$ to $y=\ell+1$, we say that the *matching* down step of U is the first instance of a down step D occurring after U which goes from $y=\ell+1$ to $y=\ell$. For example, for the Schröder path

$$P \coloneqq U_1U_2U_3HD_1U_4HHD_2HD_3D_4HU_5D_5$$

(where each up and down step is labelled by their order of occurrence), the matching down steps of U_1, U_2, U_3, U_4, and U_5 are D_4, D_3, D_1, D_2, and D_5 respectively.

Next, let $\mathcal{S}_d$ be the set of nonempty Schröder paths such that:

- the total number of H steps in the path is equal to $1+j(d-1)$ for some integer $j \geq 0$;
- between every up step U and its matching down step D, the number of H steps between U and D is equal to $1+j(d-1)$ for some integer $j \geq 0$.

We describe a simple bijection $f_1 : \mathcal{M}_d \to \mathcal{S}_d$ defined as follows. Given $M \in \mathcal{M}_d$, we scan across the expression M from left to right and:

- replace every instance of $L($ with an up step U;
- replace every indeterminate a_i with a horizontal step H;
- replace every instance of $)$ with a down step D.

For example, given $M = L(a_1L^2(a_2)a_3)a_4a_5 \in \mathcal{M}_3$, we have

$$f_1(M) = UHUUHDDHDHH,$$

which is indeed in $\mathcal{S}_3$. Figure 2 illustrates the mapping f_1 for all 21 monomials $M \in \mathcal{M}_3$ with $\operatorname{topt}(M)=3$.

Then we have the following.

Theorem 5. *Let $d \geq 2$ and $n,k \geq 0$ be integers. Then $N_d(n,k)$ counts the number of Schröder paths in $\mathcal{S}_d$ with semilength $(n-k)(d-1)+k+1$ and exactly k up steps.*

Proof. Given $M \in \mathcal{M}_d$, observe that $f_1(M)$ is indeed a Schröder path — it contains only U, H, and D steps with an equal number of U and D steps, and it cannot contain a prefix with more D steps than U steps. Additionally, every monomial has a positive degree that is congruent to 1 modulo $d-1$, which implies that the total number of horizontal steps in $f_1(M)$ is equal to $1+j(d-1)$ for some integer $j \geq 0$.

Furthermore, since the argument of an instance of L in M is a monomial in its own right, and the opening and closing parentheses of L are mapped to a pair of matching up and down steps in $f_1(M)$, we see that the number of horizontal steps between them must also be equal to $1+j(d-1)$ for some $j \geq 0$. Hence, we conclude that $f_1(M) \in \mathcal{S}_d$.

Next, observe that $|f_1(M)| = \deg(M) + \operatorname{lopt}(M)$, and there are exactly $\operatorname{lopt}(M)$ up steps in $f_1(M)$. The mapping f_1 is also clearly invertible. Thus, our claim follows. $\square$

Given a Schröder path, we call an instance of UD (i.e., an up step immediately followed by a down step) a *peak* in the path. Notice that $\mathcal{S}_2$ is simply the set of nonempty Schröder paths with no peaks, which is known to be counted by the Catalan numbers [28, Chapter 2, Exercise 45].

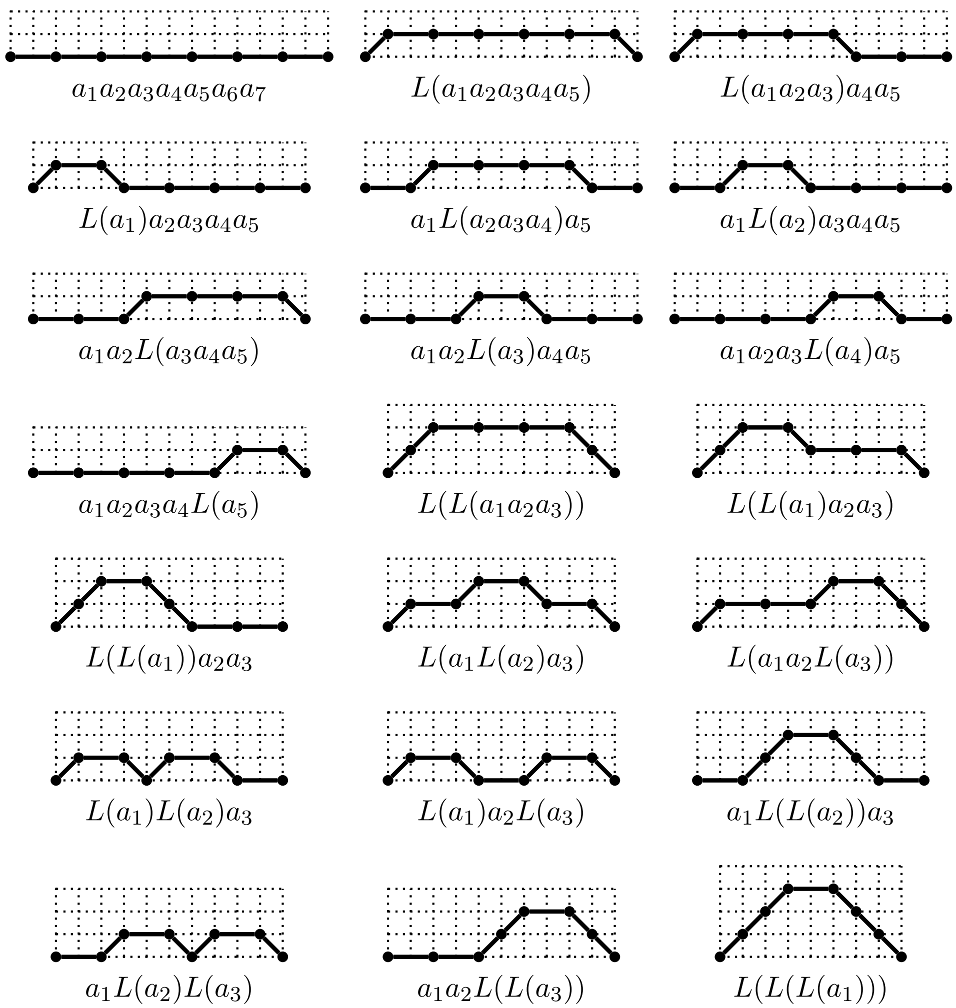

FIGURE 2. Illustrating the bijection f_1 for all $M \in \mathcal{M}_3$ with $\text{topt}(M) = 3$

3.2. **Ordered trees with restricted outdegrees.** We next show that $N_d(n,k)$ also counts a family of ordered trees (i.e., trees where the order of children at each node matters). Given an ordered tree T, the *outdegree* of a vertex $v \in T$ is the number of children of v in T. A vertex is considered a *leaf* if it has an outdegree of zero; otherwise, it is classified as an *internal node*.

Next, for an integer $d \geq 2$, let $\mathcal{T}_d$ denote the set of unlabelled ordered trees that contain at least one internal node, where the outdegree of every internal node is congruent to 1 modulo $d-1$. For instance, Figure 3 displays the 12 elements in $\mathcal{T}_3$ with 5 edges.

We next define a bijection $f_2 : \mathcal{M}_d \to \mathcal{T}_d$, as follows. Given $M \in \mathcal{M}_d$, we write $M = M_1 \cdots M_\ell$, where $M_i \in \overline{\mathcal{M}_d}$ is irreducible for each $i \in [\ell]$. Then $f_2(M)$ is defined recursively: The root node has outdegree ℓ, with subtrees $T_1, \ldots, T_\ell$ arranged from left to right. For each $i \in [\ell]$, if M_i is an indeterminate, then T_i is defined to be a one-node tree; otherwise, $M_i = L(M_i')$ for some $M_i' \in \mathcal{M}_d$, and we define $T_i \coloneqq f_2(M_i')$.

For example, Figure 4 illustrates the mapping f_2 for all 21 ternary operator monomials $M \in \mathcal{M}_3$ with $\text{topt}(M) = 3$.

Then we have the following.

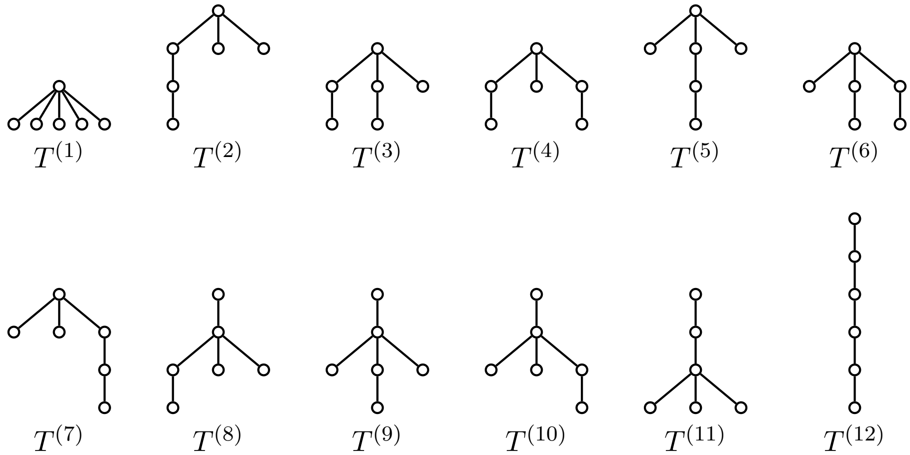

FIGURE 3. The 12 trees in $\mathcal{T}_3$ with 5 edges

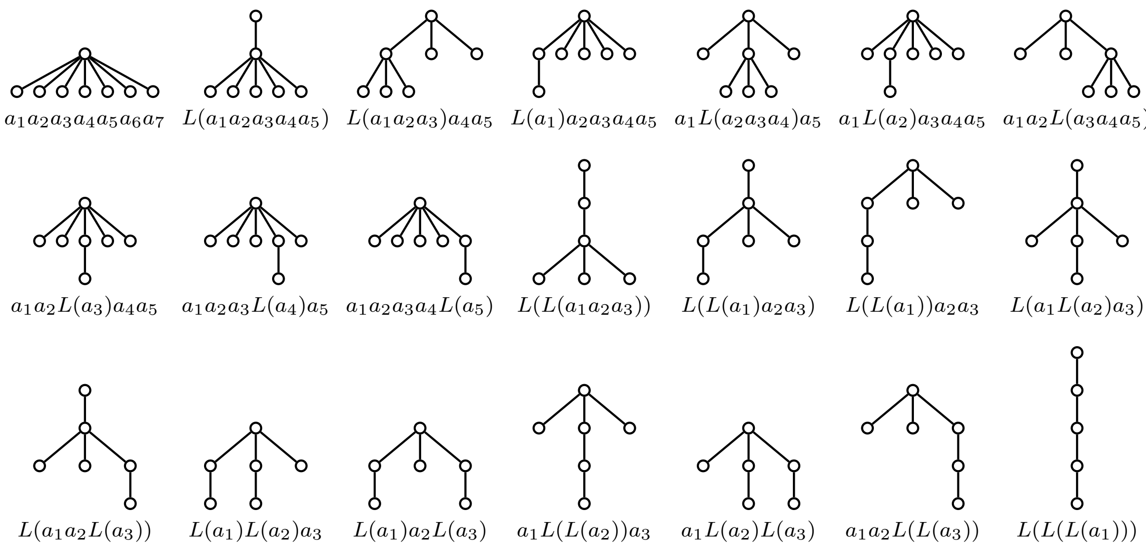

FIGURE 4. Illustrating the bijection f_2 for all $M \in \mathcal{M}_3$ with $\operatorname{topt}(M) = 3$

Theorem 6. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then $N_d(n,k)$ counts the number of trees in $\mathcal{T}_d$ with $(n-k)(d-1)+k+1$ edges and $k+1$ internal nodes.*

Proof. Observe that when $M \in \mathcal{M}_d$ is written as a product of ℓ irreducible operator monomials, we have that ℓ is congruent to 1 modulo $d-1$. Thus, every internal node in $f_2(M)$ has outdegree congruent to 1 modulo $d-1$, and hence $f_2(M) \in \mathcal{T}_d$. Next, each non-root internal node in $f_2(M)$ corresponds to an application of L in M, and thus the number of internal nodes in $f_2(M)$ is $\operatorname{lopt}(M)+1$. Moreover, each leaf of $f_2(M)$ corresponds to an indeterminate in M, which implies that $f_2(M)$ has $\deg(M) + \operatorname{lopt}(M) + 1$ vertices, and thus $\deg(M) + \operatorname{lopt}(M)$ edges.

Next, we describe the inverse mapping f_2^{-1}. Given $T \in \mathcal{T}_d$, the root of T is an internal node by definition of $\mathcal{T}_d$. Let $T_1, \ldots, T_\ell$ be the subtrees of the root node of T from left to right. Define the monomial $f_2^{-1}(T)$ as $M_1 \cdots M_\ell$ such that M_i is an indeterminate if T_i consists of a single vertex, and $M_i = L(f_2^{-1}(T_i))$ otherwise. Therefore, $f_2 : \mathcal{M}_d \to \mathcal{T}_d$ is indeed a bijection, and the claim follows. $\square$

We remark that there is a more visual and non-recursive way of describing the inverse mapping f_2^{-1}. Given a tree $T \in \mathcal{T}_d$, write symbols $a_1, a_2, \ldots$ under each leaf of T from left to right. Then, for each non-root internal node of T, write $L($ to the left of the node and $)$ to the right of the node. Finally, one can write out the monomial $f_2^{-1}(T)$ by starting at the top of the root node of T and then "walking" counterclockwise around the tree and picking up the written symbols in order. For example, for the tree in $T \in \mathcal{T}_3$ given in Figure 5, we have

$$f_2^{-1}(T) = L(L(a_1)a_2L(a_3a_4a_5)).$$

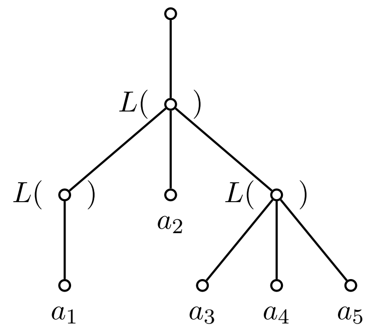

FIGURE 5. Illustrating the inverse mapping f_2^{-1}

Notice that when $d = 2$, $\mathcal{T}_d$ is simply the set of all nonempty ordered trees. Thus, Theorem 6 specializes to the fact that the Narayana number $N_2(n,k)$ counts the number of ordered trees with $n+1$ edges and $k+1$ internal nodes (see, for instance, A001263).

3.3. **Dyck paths with restricted ascents.** Now that we have a combinatorial interpretation of $N_d(n,k)$ in terms of a particular subset of ordered trees, we can leverage known bijections between classic combinatorial objects to obtain other interpretations of $N_d(n,k)$. For the remainder of the section, we will detail two such interpretations.

First, a *Dyck path* is a Schröder path which does not contain any horizontal steps (i.e., it uses only up and down steps). Given an integer $d \geq 2$, let $\mathcal{Q}_d$ denote the set of nonempty Dyck paths in which every *ascent* (i.e., a maximal subsequence of up steps in the path) has length congruent to 1 modulo $d-1$. For example, Figure 6 lists the 12 paths in $\mathcal{Q}_3$ with semilength 5.

Next, we describe the bijection between rooted ordered trees and Dyck paths. Given a tree T, let $v_1, \ldots, v_n$ be the list of vertices in T in preorder traversal (also known as depth-first traversal). Then let j_i be the outdegree of v_i for every $i \in [n]$. Observe that v_n is always a leaf, and thus j_n must be zero. We define

$$f_3(T) \coloneqq U^{j_1}DU^{j_2}D \cdots U^{j_{n-1}}D.$$

For example, for the trees from Figure 4 and Dyck paths from Figure 6, we have $f_3(T^{(i)}) = Q^{(i)}$ for every $i \in [12]$. Then we have the following result.

Theorem 7. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then $N_d(n,k)$ counts the number of Dyck paths in $\mathcal{Q}_d$ with semilength $(d-1)(n-k)+k+1$ and with exactly $k+1$ peaks.*

Proof. Given $T \in \mathcal{T}_d$, observe that every ascent in $f_3(T)$ uniquely corresponds to an internal node in T. Since every nonzero outdegree in T is congruent to 1 modulo $(d-1)$, so is the length of every ascent in $f_3(T)$, and hence $f_3(T) \in \mathcal{Q}_d$. We also obtain that $|f_3(T)|$ is one less than the number of vertices in T, which is equal to the number of edges in T. Moreover, since the number

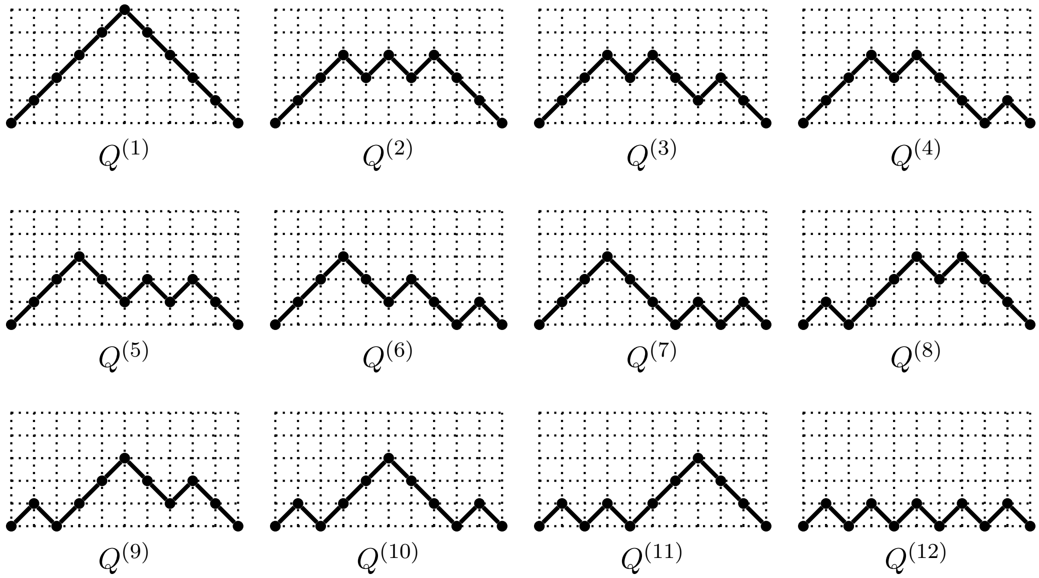

FIGURE 6. The 12 Dyck paths in $\mathcal{Q}_3$ with semilength 5

$P^{(1)} := 54321$	$P^{(2)} := 54123$	$P^{(3)} := 53124$	$P^{(4)} := 43125$
$P^{(5)} := 52134$	$P^{(6)} := 42135$	$P^{(7)} := 32145$	$P^{(8)} := 15423$
$P^{(9)} := 15324$	$P^{(10)} := 14325$	$P^{(11)} := 12543$	$P^{(12)} := 12345$

TABLE 4. The 12 permutations of $[5]$ in $\mathcal{P}_3$

of peaks in a Dyck path is equal to its number of ascents, the number of peaks in $f_3(T)$ is equal to the number of internal nodes in T.

Conversely, given a Dyck path $Q \in \mathcal{Q}_d$, we can extract $j_1 \ldots, j_{n-1}$ and recover the tree $f_3^{-1}(Q) \in \mathcal{T}_d$. Hence, $f_3 : \mathcal{T}_d \to \mathcal{Q}_d$ is a bijection, which completes the proof. $\square$

Observe that $\mathcal{Q}_2$ is the set of all nonempty Dyck paths. Thus, when $d = 2$, Theorem 7 specializes to the well-known fact that $N_2(n, k)$ counts the number of Dyck paths with semilength $n + 1$ and with exactly $k + 1$ peaks [27, Section 2.4.2].

3.4. 231**-avoiding permutations with restricted decreasing runs.** Let $P\colon [n] \to [n]$ be a permutation of $[n]$. For convenience, we will often write $P = P_1 \cdots P_n$ to indicate that the permutation P maps i to P_i for every $i \in [n]$. Also, let $\pi = \pi_1 \cdots \pi_k$ be a permutation of $[k]$ where $k \leq n$. We say that P *contains* the pattern π if there exist indices $1 \leq i_1 < i_2 < \cdots < i_k$ such that, for every distinct $j, j' \in [k]$, $P_{i_j} < P_{i_{j'}}$ if and only if $\pi_j < \pi_{j'}$. For example, $P = 43521$ contains (four instances of) the pattern $\pi = 231$. Conversely, we say that P *avoids* π if P does not contain π. Also, a *decreasing run* in a permutation P is a maximal decreasing subsequence of P. For example, $P = 314652$ contains three decreasing runs: 31, 4, and 652.

Given an integer $d \geq 2$, we let $\mathcal{P}_d$ denote the set of nonempty 231-avoiding permutations where every decreasing run has length congruent to 1 modulo $(d-1)$. For example, Table 4 lists the 12 permutations of $[5]$ in $\mathcal{P}_3$.

Recall the notion of the matching down step of a particular up step in a Schröder path, which also applies to Dyck paths. Now consider the function $f_4 : \mathcal{Q}_d \to \mathcal{P}_d$ defined as follows. Given a Dyck path Q of semilength n, define the permutation $P = f_4(Q)$ on $[n]$ such that $P_i = j$ if and only if the j-th down step in Q matches with the i-th up step in Q. For example, consider the Dyck path

$$Q = U_1U_2U_3D_1D_2U_4D_3D_4U_5U_6D_5D_6,$$

where each up and down step is labelled by the order they appear in Q. Notice that D_1 is the matching down step of U_3, and thus the image of Q under f_4 would map 3 to 1. In fact, one can check that $f_4(Q) = 421365$. Also, for the Dyck paths in Figure 6 and the permutations in Table 4, we have $f_4(Q^{(i)}) = P^{(i)}$ for every $i \in [12]$. Then we have the following.

Theorem 8. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then $N_d(n, k)$ counts the number of permutations of $[(d-1)(n-k)+k+1]$ in $\mathcal{P}_d$ with exactly $k+1$ decreasing runs.*

Proof. It is known that f_4 gives a bijection between Dyck paths and 231-avoiding permutations (see, for instance, [27, Section 2.4.3] for the details). Moreover, notice that there is a one-to-one correspondence between ascents in a Dyck path Q and decreasing runs in $f_4(Q)$. Thus, given $Q \in \mathcal{Q}_d$, every decreasing run in $f_4(Q)$ has length congruent to 1 modulo $(d-1)$, and thus $f_4(Q) \in \mathcal{P}_d$. Furthermore, we see that $f_4(Q)$ is a permutation of $[|Q|]$ and has as many decreasing runs as Q has ascents. Thus, our claim follows. □

Observe that $\mathcal{P}_2$ is the set of all nonempty 231-avoiding permutations. Thus, Theorem 8 extends the well-known fact that $N_2(n, k)$ gives the number of 231-avoiding permutations of $[n+1]$ with $k+1$ decreasing runs [27, Section 2.4.3]. Furthermore, one can use known bijections between 231-avoiding permutations and other classic combinatorial objects (such as non-crossing partitions, full binary trees, and standard Young tableaux — see, for instance, [27, Chapter 2]) to obtain yet more combinatorial interpretations of $N_d(n, k)$.

4. Several "replicative" combinatorial interpretations of $N_d(n, k)$

In this section, we describe four more combinatorial interpretations of $N_d(n, k)$. All four sets of combinatorial objects can be seen as generalizations of objects shown in Huh et al. [21] to be counted by $N_3(n, k)$.

4.1. **Schröder paths with labelled descents.** Given an integer $d \geq 2$, let $\widetilde{\mathcal{S}}_d$ denote the set of Schröder paths in which:

- every descent has length at most $d-1$;
- every descent of length $\ell \geq 1$ is labelled by an $(\ell-1)$-element subset of $[d-2]$.

For example, an element in $\widetilde{\mathcal{S}}_4$ is

$$UUUHH\underbrace{D}_{\emptyset}U\underbrace{DD}_{\{1\}}HUU\underbrace{DDD}_{\{1,2\}}H\underbrace{D}_{\emptyset}.$$

Observe that descents of length 1 are always labelled by the empty set, and when $d = 3$, descents of length 2 are always labelled by the set $\{1\}$. While it can be practically convenient to consider these descents unlabelled, we will preserve these labels to help streamline our description of a bijection from $\mathcal{M}_d$ to $\widetilde{\mathcal{S}}_d$. Given $M \in \mathcal{M}_d$, we define $f_5(M) \in \widetilde{\mathcal{S}}_d$ recursively as follows:

- If M is a single indeterminate, then $f_5(M)$ is the empty path.
- If $M = L(M')$ for some monomial $M' \in \mathcal{M}_d$, then $f_5(M) \coloneqq f_5(M')H$.
- Otherwise, we can write $M = M_0M_1M_2 \cdots M_{d-1}$ such that $M_0 \in \mathcal{M}_d$, and $M_i \in \overline{\mathcal{M}_d}$ for every $i \in [d-1]$. Let $I \coloneqq \{i \in [d-2] : \mathrm{lopt}(M_i) \geq 1\}$, and define indices $1 \leq i_1 < \cdots < i_{|I|} \leq d-2$ such that $I = \left\{i_1, \ldots, i_{|I|}\right\}$. Also, for every $i \in I$, define M_i' such that $M_i = L(M_i')$. Then we define
$$f_5(M) \coloneqq f_5(M_0)Uf_5(M_{i_1}')Uf_5(M_{i_2}') \cdots Uf_5(M_{i_{|I|}}')Uf_5(M_{d-1})\underbrace{D^{|I|+1}}_{I}.$$

For example, given $M := L(a_1 L(a_2) L^2(a_3a_4a_5)) \in \mathcal{M}_3$, we have

$$\begin{aligned} f_5(M) &= f_5(a_1 L(a_2) L^2(a_3a_4a_5)H \\ &= f_5(a_1)Uf_5(a_2)Uf_5(L^2(a_3a_4a_5))\underset{\{1\}}{\underline{DD}}H. \end{aligned}$$

Now $f_5(a_1)$ and $f_5(a_2)$ are both the empty path, while

$$f_5(L^2(a_3a_4a_5)) = f_5(a_3a_4a_5)HH = U\underset{\emptyset}{\underline{D}}HH.$$

Thus, we obtain that $f_5(M) = UUU\underset{\emptyset}{\underline{D}}HH\underset{\{1\}}{\underline{DD}}H$. Observe that $f_5(M)$ has semilength 6, which is equal to $\operatorname{topt}(M)$. Also, $f_5(M)$ has 3 instances of H and 1 instance of DD, which sum to $\operatorname{lopt}(M) = 4$. Figure 7 lists the Schröder paths $f_5(M)$ for all 21 monomials $M \in \mathcal{M}_3$ with $\operatorname{topt}(M) = 3$. Again, since every descent of length 1 and 2 are labelled by $\emptyset$ and $\{1\}$ respectively in this case, we have suppressed these labels in Figure 7 to reduce cluttering.

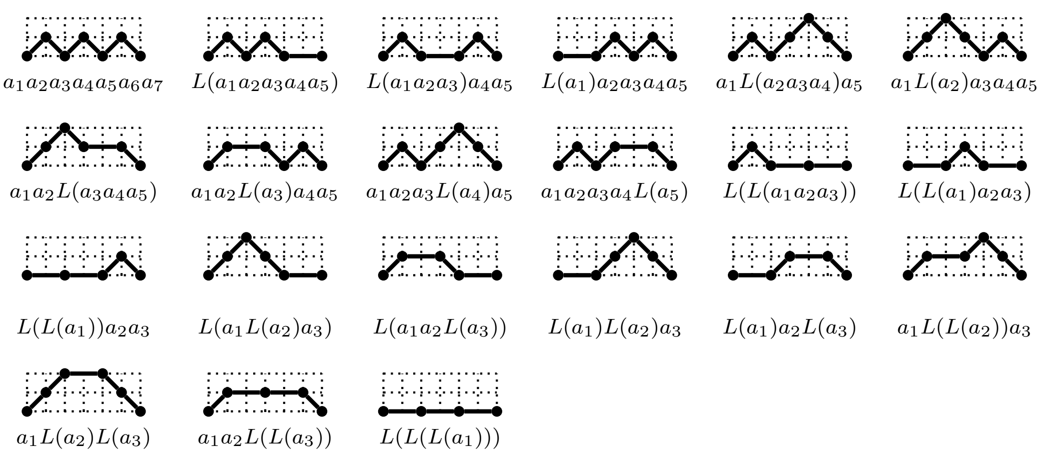

FIGURE 7. Illustrating the bijection f_5 for all $M \in \mathcal{M}_3$ with $\operatorname{topt}(M) = 3$ (with descent labels suppressed)

Theorem 9 shows that $\widetilde{\mathcal{S}}_d$ provides yet another combinatorial interpretation for $N_d(n,k)$.

Theorem 9. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then $N_d(n,k)$ counts the number of Schröder paths in $\widetilde{\mathcal{S}}_d$ with semilength n and with exactly k occurrences in total of H and DD.*

Proof. For convenience, given a Schröder path S, we let $\operatorname{hdd}(S)$ denote the total number of instances of H and DD in S. Let $M \in \mathcal{M}_d$. We examine the mapping f_5 case by case.

If M is the single indeterminate, then $\operatorname{topt}(M) = \operatorname{lopt}(M) = 0$, and $f_5(M)$, the empty path, also has $|f_5(M)| = \operatorname{hdd}(f_5(M)) = 0$.

If $M = L(M')$ for some $M' \in \mathcal{M}_d$, then $\operatorname{topt}(M') = \operatorname{topt}(M) - 1$ and $\operatorname{lopt}(M') = \operatorname{lopt}(M) - 1$. Now $f_5(M) = f_5(M')H$, and so we also have $|f_5(M')| = |f_5(M)| - 1$ and $\operatorname{hdd}(f_5(M')) = \operatorname{hdd}(f_5(M)) - 1$.

Now suppose $M = M_0M_1 \cdots M_{d-1}$ where M_i is irreducible for every $i \in [d-1]$. From the definition of f_5, we have

$$f_5(M) = f_5(M_0)Uf_5(M'_{i_1})Uf_5(M'_{i_2}) \cdots Uf_5(M_{i_{|I|}})Uf_5(M_{d-1})\underset{I}{\underline{D^{|I|+1}}}.$$

In this case, we have

$$\operatorname{topt}(f_5(M)) = \operatorname{topt}(M_0) + \sum_{i \in I} \operatorname{topt}(M_i') + \operatorname{topt}(M_{d-1}) + 1 + |I|,$$
$$\operatorname{lopt}(f_5(M)) = \operatorname{lopt}(M_0) + \sum_{i \in I} \operatorname{lopt}(M_i') + \operatorname{lopt}(M_{d-1}) + |I|.$$

We also have $|f_5(M)| = |f_5(M_0)| + \sum_{i \in I} |f_5(M_i')| + |f_5(M_{d-1})| + 1 + |I|$. Notice that given any irreducible monomial M, $f_5(M)$ is either the empty path or must end with an H, and thus $f_5(M_{d-1})$ cannot end with a D. Therefore, $f_5(M)$ ends with a descent of length $|I|+1$, which contains $|I|$ instances of DD. Hence, $\operatorname{hdd}(f_5(M)) = \operatorname{hdd}(f_5(M_0)) + \sum_{i \in I} \operatorname{hdd}(f_5(M_i')) + \operatorname{hdd}(f_5(M_{d-1})) + |I|$.

Since f_5 never produces a descent of length at least d and labels every descent of length $\ell \geq 1$ with an $(\ell-1)$-element subset of $[d-2]$, we have $f_5(M) \in \widetilde{\mathcal{S}}_d$ for every $M \in \mathcal{M}_d$, with $|f_5(M)| = \operatorname{topt}(M)$ and $\operatorname{hdd}(f_5(M)) = \operatorname{lopt}(M)$. To complete our proof, it suffices to show that the inverse function of f_5 exists.

Let $S \in \widetilde{\mathcal{S}}_d$. If S is the empty path, then $f_5^{-1}(S)$ is the monomial with a single indeterminate. Otherwise, S must either end with an H or a D. If $S = S'H$ for some $S' \in \widetilde{\mathcal{S}}_d$, then $f_5^{-1}(S) = L(f_5^{-1}(S'))$. Otherwise, suppose S ends with a descent of length $\ell \geq 1$. Then we can uniquely write

$$S = S^{(0)} U S^{(1)} U S^{(2)} \cdots U S^{(\ell)} \underset{I}{\underline{D^{\ell}}}$$

for some $S^{(0)}, \ldots, S^{(\ell)} \in \widetilde{\mathcal{S}}_d$, with an $(\ell-1)$-element set $I \subseteq [d-2]$ marking the terminal descent of S. Then we obtain that $f_5^{-1}(S) = M_0 M_1 \cdots M_d$ where

- $M_0 = f_5^{-1}(S^{(0)})$ and $M_{d-1} = f_5^{-1}(S^{(\ell)})$.
- For every $i \in [d-2]$, if $i \notin I$, then M_i is a single indeterminate. Otherwise, if i is the j-th smallest index in I, then $M_i = L(f_5^{-1}(S^{(j)}))$.

Thus, the mapping f_5 is indeed invertible, and our claim follows. □

Again, given $S \in \widetilde{\mathcal{S}}_d$, every descent of length 1 is labelled by $\emptyset$. Thus, we can ignore these markings, in which case $\widetilde{\mathcal{S}}_2$ gives exactly the (unlabelled) Schröder paths without descents of length at least 2, which is known to be counted by the Catalan numbers [28, Chapter 2, Exercise 49]. Likewise, when $d = 3$, every instance of DD is labelled by the set $\{1\}$, and so we can ignore these markings as well. In that case, our theorem specializes to Huh et al.'s result [21, Theorem 6.2(2) and Corollary 6.3], which shows that $N_3(n,k)$ counts the number of Schröder paths S where $|S| = n$, $\operatorname{hdd}(S) = k$, and S does not contain descents of length at least 3.

4.2. **Generalized F-paths.** We next describe a set of lattice paths that can be seen as a generalization of the notion of F-paths defined in Huh et al. [21]. Given an integer $d \geq 2$, let $\mathcal{F}_d$ denote the set of lattice paths such that:

- the path begins at $(0,0)$, and always stays on or above the line $y = x$;
- every step has the form $(\ell, 1)$ for some integer $\ell \geq 0$;
- every step of the form $(\ell, 1)$ with $\ell \geq 1$ is labelled by a composition of $\ell - 1$ consisting of $d-1$ nonnegative parts (i.e., an ordered list of $d-1$ nonnegative integers which sum to $\ell - 1$).

For example,

$$F = (0,1), (0,1), \underset{(0,0)}{\underline{(1,1)}}, (0,1), \underset{(0,2)}{\underline{(3,1)}}, \underset{(1,0)}{\underline{(2,1)}}$$

is an element of $\mathcal{F}_3$. Also, given $F \in \mathcal{F}_d$, we let $|F|$ denote the number of steps in F. For instance, Figure 8 illustrates the 6 elements in $\mathcal{F}_3$ where $|F| = 2$.

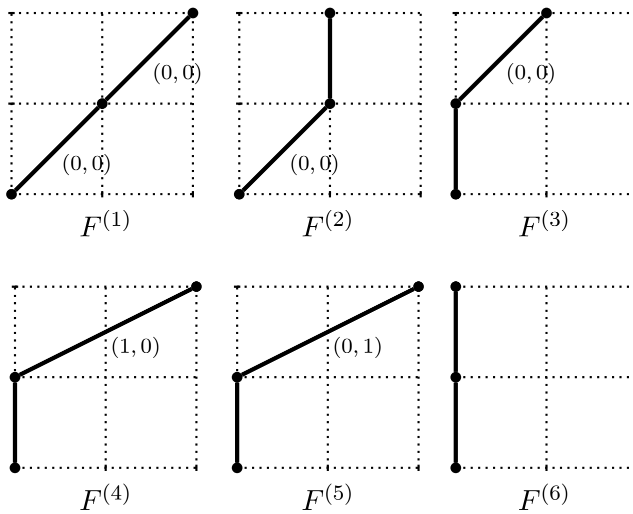

FIGURE 8. The 6 lattice paths $F \in \mathcal{F}_3$ with $|F| = 2$

Observe from the definition of $\mathcal{F}_d$ that the step $(1,1)$ is always labelled by the composition where every part is 0. As with the elements in $\widetilde{\mathcal{S}}_d$, preserving these somewhat redundant markings will be helpful in establishing our results.

Next, given $M \in \mathcal{M}_d$, we define $\operatorname{lofi}(M)$ (*linear operators on first indeterminate*) to be the number of linear operators L applied to the first (i.e., leftmost) indeterminate in M. For instance, among $M \in \mathcal{M}_3$ with $\operatorname{topt}(M) = 3$, the seven elements with $\operatorname{lofi}(M) = 1$ are

$$\begin{array}{llll} L(a_1a_2a_3a_4a_5), & L(a_1a_2a_3)a_4a_5, & L(a_1)a_2a_3a_4a_5, & L(a_1(L(a_2)a_3)), \\ L(a_1a_2L(a_3)), & L(a_1)L(a_2)a_3, & L(a_1)a_2L(a_3). & \end{array}$$

Observe that $0 \leq \operatorname{lofi}(M) \leq \operatorname{lopt}(M)$ for every $M \in \mathcal{M}_d$. Next, given $F \in \mathcal{F}_d$, let $\operatorname{north}(F)$ denote the number of $(0,1)$ steps in F. Also, let $\operatorname{ht}(F)$ denote $y_n - x_n$, where (x_n, y_n) is the terminal point of the path F. For example, for the paths in Figure 8, we have

$$\operatorname{ht}(F^{(1)}) = \operatorname{ht}(F^{(4)}) = \operatorname{ht}(F^{(5)}) = 0, \quad \operatorname{ht}(F^{(2)}) = \operatorname{ht}(F^{(3)}) = 1, \quad \operatorname{ht}(F^{(6)}) = 2.$$

Next, we define the mapping $f_6 : \mathcal{M}_d \to \mathcal{F}_d$ recursively as follows.

- If M is a single indeterminate, then $f_6(M)$ is the empty path.
- If $M = L(M')$ for some monomial $M' \in \mathcal{M}_d$, then $f_6(M) \coloneqq f_6(M'), (0,1)$.
- Otherwise, we can write $M = M_0M_1 \cdots M_{d-1}$ such that $M_0 \in \mathcal{M}_d$ and $M_i \in \overline{\mathcal{M}_d}$ for every $i \in [d-1]$. Let $m_i \coloneqq \operatorname{lofi}(M_i)$ for every $i \in [d-1]$, and let $m \coloneqq 1 + \sum_{i=1}^{d-1} m_i$. Define

$$f_6(M) \coloneqq F_0, F_1, \ldots, F_{d-1}, \quad \underset{(m_1,\ldots,m_{d-1})}{\underline{(m,1)}}$$

  where $F_0 \coloneqq f_6(M_0)$. For every $i \in [d-1]$, F_i is the empty path if M_i is a single indeterminate; otherwise $M_i = L(M_i')$ for some $M_i' \in \mathcal{M}_d$, and we define $F_i \coloneqq (0,1), f_6(M_i')$.

For example, let $M := L(a_1L(a_2a_3a_4)L^2(a_5)) \in \mathcal{M}_3$. Then $f_6(M)$ is computed as follows.

$$\begin{aligned} f_6(M) &= f_6(L(a_1L(a_2a_3a_4)L^2(a_5))) \\ &= f_6(a_1L(a_2a_3a_4)L^2(a_5)), (0,1) \\ &= (0,1), f_6(a_2a_3a_4), (0,1), f_6(L(a_5)), \underset{(1,2)}{\underline{(4,1)}}, (0,1) \\ &= (0,1), \underset{(0,0)}{\underline{(1,1)}}, (0,1), (0,1), \underset{(1,2)}{\underline{(4,1)}}, (0,1), \end{aligned}$$

which is indeed an element in $\mathcal{F}_3$. Notice that $|f_6(M)| = \mathrm{topt}(M) = 6$, $\mathrm{north}(f_6(M)) = \mathrm{lopt}(M) = 4$, and $\mathrm{ht}(f_6(M)) = \mathrm{lofi}(M) = 1$. Also, for the paths given in Figure 8, we have

$$\begin{aligned} &f_6(a_1a_2a_3a_4a_5) = F^{(1)}, \quad f_6(L(a_1a_2a_3)) = F^{(2)}, \quad f_6(L(a_1)a_2a_3) = F^{(3)}, \\ &f_6(a_1L(a_2)a_3) = F^{(4)}, \quad\ \ f_6(a_1a_2L(a_3)) = F^{(5)}, \quad f_6(L(L(a_1))) = F^{(6)}. \end{aligned}$$

Next, we demonstrate that f_6 is indeed a bijection.

Theorem 10. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then $N_d(n,k)$ counts the number of lattice paths in $\mathcal{F}_d$ consisting of a total of n steps, k of which are $(0,1)$.*

Proof. Let $M \in \mathcal{M}_d$. Notice that every step in $f_6(M)$ has the form $(\ell, 1)$ for some $\ell \geq 0$, and by construction $f_6(M)$ always stays on or above the line $y = x$. Also, when $\ell \geq 1$, f_6 labels the step $(\ell, 1)$ with a composition of $\ell - 1$ with $d-1$ nonnegative parts. Thus, $f_6(M) \in \mathcal{F}_d$ for every $M \in \mathcal{M}_d$.

Next, we prove that $|f_6(M)| = \mathrm{topt}(M)$ by strong induction on $\mathrm{topt}(M)$. If $\mathrm{topt}(M) = 0$, then M is a single indeterminate, and $f_6(M)$ is the empty path, leading to $|f_6(M)| = 0 = \mathrm{topt}(M)$. Now suppose $\mathrm{topt}(M) \geq 1$. If $M = L(M')$ for some $M' \in \mathcal{M}_d$, then $\mathrm{topt}(M') = \mathrm{topt}(M) - 1$, and by the inductive hypothesis, $|f_6(M')| = \mathrm{topt}(M')$. Therefore, we have $f_6(M) = f_6(M'), (0,1)$, which implies

$$|f_6(M)| = |f_6(M')| + 1 = \mathrm{topt}(M') + 1 = \mathrm{topt}(M).$$

Otherwise, we can express M as $M = M_0M_1 \cdots M_{d-1}$, where M_i is irreducible for every $i \in [d-1]$. Notice that $\mathrm{topt}(M) = \sum_{i=1}^{d} \mathrm{topt}(M_i) + 1$. From the definition of f_6, we have:

$$f_6(M) = F_0, F_1, \ldots, F_{d-1}, \underset{(m_1,\ldots,m_{d-1})}{\underline{(m,1)}}.$$

For each $i \in [d-1]$, either F_i is the empty path (in which case $|F_i| = \mathrm{topt}(M_i) = 0$), or $F_i = (0,1), f_6(M_i')$ where $M_i = L(M_i')$. In the latter case, we know that $|f_6(M_i')| = \mathrm{topt}(M_i')$ by the inductive hypothesis, and so we have

$$|F_i| = |f_6(M_i')| + 1 = |M_i'| + 1 = |M_i|.$$

Additionally, since F_0 is defined as $f_6(M_0)$, we conclude that $|F_i| = \mathrm{topt}(M_i)$ for every $i \in \{0, \ldots, d-1\}$. Hence,

$$|f_6(M)| = \sum_{i=0}^{d-1} |F_i| + 1 = \sum_{i=0}^{d-1} \mathrm{topt}(M_i) + 1 = \mathrm{topt}(M).$$

This shows that $|f_6(M)| = \mathrm{topt}(M)$ in all cases. The same inductive argument can be applied to show that $\mathrm{north}(f_6(M)) = \mathrm{lopt}(M)$ and $\mathrm{ht}(f_6(M)) = \mathrm{lofi}(M)$ for every $M \in \mathcal{M}_d$.

Next, we describe the inverse function of f_6 to complete the proof. Let $F \in \mathcal{F}_d$. If F is the empty path, then $f_6^{-1}(F)$ is the monomial with a single indeterminate. Otherwise, we can express F as $F = F', (\ell, 1)$ for some $F' \in \mathcal{F}_d$ and $\ell \geq 0$. If $\ell = 0$, then we have $f_6^{-1}(F) = L(f_6^{-1}(F'))$.

If $\ell \geq 1$, the step $(\ell, 1)$ is labelled by a composition $(\ell_1, \ldots, \ell_{d-1})$ of $\ell - 1$. In this case, we note that $\operatorname{ht}(F') = \operatorname{ht}(F) + m - 1$. This implies that for every $j \in [\operatorname{ht}(F) + \ell - 2]$, there exists a $(0,1)$ step in F' that transitions from height $j-1$ to height j. Then we write

$$F' = F_0, F_1, \ldots, F_{d-1}$$

such that for every $i \in [d-1]$, F_i is the empty path if $\ell_i = 0$; otherwise, the first step of F_i is the last instance of $(0,1)$ in F' that brings the path from height $\operatorname{ht}(F) + \sum_{j=1}^{i-1} \ell_j$ to $\operatorname{ht}(F) + \sum_{j=1}^{i-1} \ell_j + 1$. By choosing to start at the last instance of $(0,1)$ at a given height, the path F_i (as a lattice path in its own right) must start with $(0,1)$ and cannot subsequently dip below the line $y = x + 1$. Thus, we can write $F_i = (0,1), F_i'$ for some $F_i' \in \mathcal{F}_d$ in this case. Moreover, notice that $\operatorname{ht}(F_i) = \ell_i$ by construction, which also implies that $\operatorname{ht}(F_0) = \operatorname{ht}(F)$.

From there, we can express $f_6^{-1}(F) = M_0 M_1 \cdots M_{d-1}$ as follows. First, $M_0 = f_6^{-1}(F_0)$. For every $i \in [d-1]$, if F_i is the empty path, then M_i is a single indeterminate. Otherwise, if $F_i = (0,1), F_i'$ for some $F_i' \in \mathcal{F}_d$, then $M_i = L(f_6^{-1}(F_i'))$. Thus, the inverse of f_6 is well-defined, completing the proof. □

We note that $\mathcal{F}_3$ is a slight variant of the F-paths studied in Huh et al. [21]. More precisely, the set of eligible steps for our lattice paths in $\mathcal{F}_3$ is given by

$$\left\{ \underset{(j, \ell-1-j)}{\underline{(\ell, 1)}} : \ell \geq 1, j \geq 0 \right\} \cup \{(0,1)\}.$$

In contrast, the set of eligible steps for the original F-paths from [21] is

$$\{(a,b) : a \geq 1, b \leq 1\} \cup \{(0,1)\}.$$

In particular, a bijection from $\mathcal{F}_3$ to the set of original F-paths can be established by replacing every step of the form $\underset{(j, \ell-1-j)}{\underline{(\ell, 1)}}$ with the step $(\ell - j, 1 - j)$.

Additionally, when $d = 2$, we observe that every step $(\ell, 1)$ (where $\ell \geq 1$) is labelled by the one-part composition $(\ell - 1)$. Therefore, we can ignore these markings in this case. Now notice that the lattice path

$$F \coloneqq (\ell_1, 1), (\ell_2, 1), \ldots, (\ell_n, 1)$$

belongs to $\mathcal{F}_2$ if and only if the following conditions hold:

- $\ell_i \geq 0$ for every $i \in [n]$, and
- $\sum_{j=1}^{i} \ell_j \leq i$ for every $i \in [n]$.

(The latter condition ensures that the path remains on or above the line $y = x$.) Now given a sequence $(\ell_1, \ldots, \ell_n)$ which satisfies the above, consider the sequence $(a_1, \ldots, a_{n+1})$ where $a_i \coloneqq 1 + \sum_{j=1}^{i-1} \ell_j$ for every $i \in [n+1]$. Then we have $1 = a_1 \leq a_2 \leq \cdots \leq a_{n+1}$, with $a_i \leq i$ for every $i \in [n+1]$, which gives a known combinatorial interpretation of the Catalan numbers (see, for instance, [28, Chapter 2, Exercise 78]).

4.3. **Labelled Dyck paths.** Given $d \geq 2$, we define $\widetilde{\mathcal{Q}}_d$ to be the set of nonempty Dyck paths such that:

- the last descent is unlabelled;
- every other descent of length $\ell \geq 1$ is labelled by a composition of $\ell - 1$ with $d - 1$ nonnegative parts.

For instance, an element in $\widetilde{\mathcal{Q}}_3$ is

$$UUU\underbrace{\underline{DD}}_{(0,1)}UU\ \underbrace{\underline{D}}_{(0,0)}\ UU\underbrace{\underline{DDDD}}_{(2,1)}UUDD.$$

Figure 9 shows all 6 labelled Dyck paths in $\widetilde{\mathcal{Q}}_3$ with semilength 3.

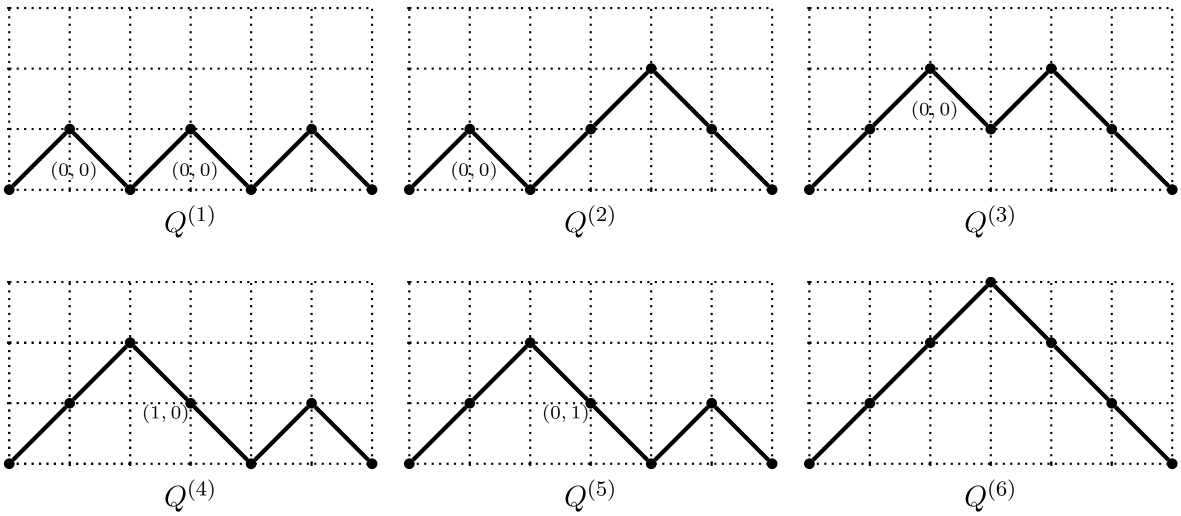

FIGURE 9. The 6 labelled Dyck paths $Q \in \widetilde{Q}_3$ with $|Q| = 3$

We show that there is a very simple bijection between $\mathcal{F}_d$ and $\widetilde{\mathcal{Q}}_d$. Given $F \in \mathcal{F}_d$ where

$$F = (\ell_1, 1), (\ell_2, 1), \ldots, (\ell_n, 1),$$

define

$$f_7(F) \coloneqq UD^{\ell_1}UD^{\ell_2} \cdots UD^{\ell_n}UD^{n+1-\sum_{i=1}^{n} \ell_i}.$$

Moreover, for every $i \in [n]$ where $\ell_i \geq 1$, we mark the descent D^{ℓ_i} in $f_7(F)$ by the same composition that labelled the step $(\ell_i, 1)$ in F. For example, consider $F \in \mathcal{F}_3$ where

$$F \coloneqq (0,1), \underbrace{\underline{(1,1)}}_{(0,0)}, (0,1), (0,1), \underbrace{\underline{(4,1)}}_{(1,2)}, (0,1).$$

Then

$$f_7(F) = UU\ \underbrace{\underline{D}}_{(0,0)}\ UUU\underbrace{\underline{DDDD}}_{(1,2)}UUDD.$$

Also, for the lattice paths in Figure 8 and labelled Dyck paths in Figure 9, we have $f_7(F^{(i)}) = Q^{(i)}$ for every $i \in [6]$. Then we have the following.

Theorem 11. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then $N_d(n, k)$ counts the number of labelled Dyck paths in $\widetilde{\mathcal{Q}}_d$ with semilength $n + 1$ and k instances of UU.*

Proof. Given $F \in \mathcal{F}_d$, where

$$F = (\ell_1, 1), (\ell_2, 1), \ldots, (\ell_n, 1),$$

we know by the definition of $\mathcal{F}_d$ that $\sum_{i=1}^{j} \ell_i \leq j$ for every $j \in [n]$, which assures that $f_7(F)$ never contains a prefix with more down steps than up steps. Also, given that F contains n steps, $f_7(F)$ must contain exactly $n + 1$ up steps and $n + 1$ down steps, and thus must indeed be a Dyck path with semilength $n + 1$. Thus, $|f_7(F)| = |F| + 1$. Also, observe that the i-th instance of U in $f_7(F)$ is followed by another U if and only if $\ell_i = 0$. Thus, we see that the number of instances of UU in $f_7(F)$ is exactly equal to the number of instances of $(0,1)$ in F. Furthermore, since the composition labellings are unaltered by f_7, we see that $f_7(F) \in \widetilde{\mathcal{Q}}_d$ for every $F \in \mathcal{F}_d$. Finally, it is rather straightforward to see that f_7 is reversible, and thus the claim follows. $\square$

We remark that, instead of labelling each descent of length ℓ with a composition $(\ell_1, \ldots, \ell_{d-1})$ which sums to $\ell - 1$, we can equivalently think of allowing $d-1$ possible types of down steps $D_1, \ldots, D_{d-1}$, and requiring that each descent (except the last one in the path) of length ℓ has the form

$$D_1^{\ell_1} D_2^{\ell_2} \cdots D_{d-1}^{\ell_{d-1}+1}.$$

This gives a $(d-1)$-coloured Dyck path interpretation of $\widetilde{\mathcal{Q}}_d$. When $d = 3$, this specializes to the restricted bi-coloured Dyck paths studied in Bényi et al. [7], Huh et al. [21], and Yan and Lin [31].

Moreover, when $d = 2$, there is only $d - 1 = 1$ type of down steps, and $\widetilde{\mathcal{Q}}_2$ is simply the set of (unlabelled and nonempty) Dyck paths. In this case, Theorem 11 assures that $N_2(n,k)$ counts the number of Dyck paths with semilength $n+1$ and k instances of UU. While we cannot find a reference mentioning this exact fact, we offer a simple independent proof of it using known properties about Narayana numbers and Dyck paths.

Corollary 12. *Let $n, k \geq 0$ be integers. Then $N_2(n,k)$ counts the number of Dyck paths with semilength $n+1$ and k instances of UU.*

Proof. Recall that $N_2(n,k)$ counts the number of Dyck paths with semilength $n+1$ and $k+1$ peaks [27, Section 2.4.2]. Also, in any Dyck path Q, every up step is either followed by another up step (which creates an instance of UU) or a down step (which forms a peak). Thus, the number of UU and peak instances must sum to $|Q|$. Hence, we see that $N_2(n,k)$ counts the number of Dyck paths with semilength $n+1$ and $n-k$ instances of UU. Finally, it is easy to see from the formula of $N_2(n,k)$ that $N_2(n,k) = N_2(n, n-k)$ for every $n, k \geq 0$. Thus, the claim follows. $\square$

Since $N_d(n,k) \neq N_d(n, n-k)$ in general, the proof of Corollary 12 seemingly cannot be extended to translate Theorem 11 into counting paths in $\widetilde{\mathcal{Q}}_d$ in terms of the number of peaks when $d \geq 3$.

4.4. **Labelled ordered trees.** As is the case for the mapping $f_3 : \mathcal{T}_d \to \mathcal{Q}_d$ discussed in Section 3.3, herein we leverage similar ideas to obtain a labelled ordered tree interpretation of $N_d(n,k)$ by applying a path-to-tree mapping from $\widetilde{\mathcal{Q}}_d$. Given $d \geq 2$, we define $\widetilde{\mathcal{T}}_d$ to be the set of labelled ordered trees such that:

- the root node is an internal node, and is unlabelled;
- every non-root internal node with outdegree $\ell \geq 1$ is labelled by a composition of $\ell - 1$ with $d-1$ nonnegative parts.

Figure 10 illustrates the 6 trees in $\widetilde{\mathcal{T}}_d$ with 3 edges.

We define a bijection $f_8 : \widetilde{\mathcal{Q}}_d \to \widetilde{\mathcal{T}}_d$ as follows. Given $Q \in \widetilde{\mathcal{Q}}_d$ with semilength n, find integers $\ell_1, \ldots, \ell_n$ where

$$Q = UD^{\ell_1} UD^{\ell_2} \cdots UD^{\ell_n}.$$

Then we let $f_8(Q)$ be the unique ordered tree on $n+1$ vertices for which the sequence of vertex outdegrees read in preorder traversal is $(\ell_n, \ldots, \ell_1, 0)$. Moreover, for every $i \in [n-1]$, the non-root internal node in $f_8(Q)$ with outdegree ℓ_i is labelled by the same composition that labelled the descent D^{ℓ_i} in Q. For example, consider the labelled Dyck paths in Figure 9 and labelled ordered trees in Figure 10. Then $f_8(Q^{(i)}) = T^{(i)}$ for every $i \in [6]$. Then we have the following.

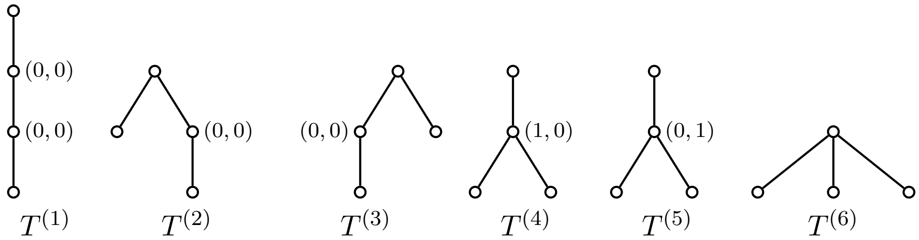

FIGURE 10. The 6 trees in $\widetilde{\mathcal{T}}_3$ with 3 edges

Theorem 13. *Let $d \geq 2$ and $n, k \geq 0$ be integers. Then $N_d(n, k)$ counts the number of labelled ordered trees in $\widetilde{\mathcal{T}}_d$ with $n+1$ edges and $k+1$ leaves.*

Proof. Suppose we are given $Q := UD^{\ell_1}UD^{\ell_2}\cdots UD^{\ell_n} \in \widetilde{\mathcal{Q}}_d$ where, for every $i \in [n-1]$, the descent D^{ℓ_i} is labelled by a composition of $\ell - 1$ with $d-1$ nonnegative parts. By construction, $f_8(Q)$ is a tree whose root node has outdegree ℓ_n. Since the last descent in Q is unlabelled, so is the root node of $f_8(Q)$. Also, since $f_8(Q)$ has $|Q|+1$ vertices, it must have $|Q|$ edges. Next, the number of leaves in $f_8(Q)$ is equal to the number of zero entries in the sequence $(\ell_n, \ldots, \ell_1, 0)$, which in turn is equal to the number of instances of $(0, 1)$ in Q plus 1.

The inverse mapping of f_8 is also straightforward. Given $T \in \widetilde{\mathcal{T}}_d$, one lists the outdegrees of its vertices in preorder traversal, omitting the last entry (which must be zero), and then apply the path-to-tree mapping f_3 defined in Section 3.3. This results in a Dyck path Q' where the first ascent is unmarked, while every other ascent of length ℓ is labelled by a composition of $\ell-1$ with $d-1$ parts. Then we reflect Q' horizontally by swapping its up and down steps and reading the steps in reverse order, which results in the labelled Dyck path $f_8^{-1}(T) \in \widetilde{\mathcal{Q}}_d$. □

Notice that when $d = 3$, each non-root internal node with outdegree $\ell \geq 1$ has ℓ possible labels: $(0, \ell-1), (1, \ell-2), \ldots, (\ell-1, 0)$. Thus, $\mathcal{F}_3$ can be alternatively seen as the set of ordered trees in which every non-root internal node is labelled by a positive integer less than or equal to its outdegree. This set of trees has been studied in Bényi et al. [7], Huh et al. [21], and Yan and Lin [31]. When $d = 2$, the marking of each internal node is determined by its outdegree, and thus we might consider the trees in $\widetilde{\mathcal{T}}_2$ unlabelled. Thus, as with Theorem 7, Theorem 13 implies that $N_2(n, k)$ counts the number of unlabelled ordered trees with $n+1$ edges and $k+1$ leaves.

5. Some future research directions

In this manuscript, we introduced the generalized Narayana numbers $N_d(n, k)$, and saw nine combinatorial interpretations of them, some of which generalize known interpretations of $N_3(n, k)$. On the other hand, there are a number of other combinatorial objects which correspond to $C_3(n)$ and $N_3(n, k)$ that we have not discussed. For instance, it is known that $C_3(n)$ counts the following sets:

- permutations of $[n+1]$ which avoid the patterns 4123, 4132, and 4213 [1];
- permutations of $[n+1]$ which avoid the patterns 2341, 2431, and 3241 [14, 17, 21];
- inversion sequences of length $n+1$ (i.e., $(\ell_1, \ldots, \ell_{n+1})$ where $0 \leq \ell_i \leq i-1$ for every $i \in [n+1]$) which avoid the patterns 101 and 102 [15, 21];
- inversion sequences of length $n+1$ which avoid the patterns 101 and 021 [21, 31].

Can any of these results be generalized to objects counted by $C_d(n)$ and $N_d(n, k)$? Since $C_2(n)$ and $C_3(n)$ are known to count certain subsets of permutations on $[n+1]$, it readily follows that $C_d(n) \leq (n+1)!$ holds for all $n \geq 0$ when $d \in \{2, 3\}$. However, the inequality fails starting at

$d = 4$ (e.g., $C_4(2) = 7 > 3!$). Thus, if a generalization exists, it would likely involve permutations or inversion sequences of size greater than $n + 1$, and/or the use of differentiating labellings.

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