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Notes and note pairs in Nørgård's infinity series

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The Danish composer Per Nørgård (born July 13 1932) defined the "infinity series" $\mathbf{s} = (s(n))_{n\geq 0}$ by the rules $s(0) = 0$, $s(2n) = -s(n)$ for $n \geq 1$, and $s(2n+1) = s(n)+1$ for $n \geq 0$; it figures prominently in many of his compositions. Here we give several new results about this sequence: first, the set of binary representations of the positions of each note forms a context-free language that is not regular; second, a complete characterization of exactly which note pairs appear; third, consecutive occurrences of identical phrases are widely separated. We also consider to what extent the infinity series is unique.

Keywords: Nørgård's infinity series; hierarchical music; note pair; Pascal's triangle; fixed point; context-free language; binomial coefficient; running sum; interval; repetition; self-similarity

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1. Introduction

The Danish composer Per Nørgård constructed an infinite sequence of integers, $\mathbf{s} = (s(n))_{n\geq 0}$, which he called the *Uendelighedsrækken* or "infinity series," using the rules

$$s(n) = \begin{cases} 0, & \text{if } n = 0; \\ -s(n/2), & \text{if } n \text{ is even;} \\ s(\frac{n-1}{2}) + 1, & \text{if } n \text{ is odd.} \end{cases} \qquad (1)$$

The nth term of Nørgård's sequence is $s(n)$; it specifies how far away the nth note is, in half steps, from some fixed base note (such as $G = 0$). The first note is $s(0) = 0$. A positive value of $s(n)$ represents a note of higher pitch than the base note $s(0)$, and a negative number represents a note of lower pitch. Figure 1 gives the first 32 notes of Nørgård's sequence starting at the note G.

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Figure 1. The first 32 notes of the infinity series starting on G.

The following table gives the first 16 terms of Nørgård's sequence, which may be compared to Figure 1. It is sequence A004718 in the *On-Line Encyclopedia of Integer Sequences* (Sloane 2016).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$s(n)$	0	1	−1	2	1	0	−2	3	−1	2	0	1	2	−1	−3	4	1	0	−2	3

Table 1. The first few terms of Nørgård's *Uendelighedsrækken* sequence.

Per Nørgård's music has often been described as "hierarchical," and the mathematical definition of Nørgård's sequence given above illustrates this self-similarity: by taking every second note starting with the note 0, we get the original melody inverted (i.e., negated), and by taking every second note starting with note 1, we get the original melody transposed up by one half step. Even more is true, as noted in (Mortensen 2003b):

THEOREM 1.1 *Every subsequence* $(s(2^e n + i))_{n \geq 0}$*, for* $e \geq 0$ *and* $0 \leq i < 2^e$*, is either the original sequence* $(s(n))_{n \geq 0}$ *transposed, or the original sequence inverted and then transposed.*

Proof. We proceed by induction on e. The result is trivial for $e = 0$. Now assume it is true for all $e' < e$; we prove it for $e \geq 1$. If i is even, then $s(2^e n + i) = s(2(2^{e-1}n + i/2)) = -s(2^{e-1}n + i/2)$. If i is odd, then $s(2^e n + i) = s(2(2^{e-1}n + (i-1)/2) + 1) = s(2^{e-1}n + (i-1)/2)$. But by induction both subsequences $(s(2^{e-1}n + i/2))_{n \geq 0}$ and $(s(2^{e-1}n + (i-1)/2))_{n \geq 0}$ are of the form $(-s(n) + c)_{n \geq 0}$ or $(s(n) + c)_{n \geq 0}$ for some constant c. Hence so is $(s(2^e n + i))_{n \geq 0}$. ■

The infinity series figures prominently in many of Nørgård's compositions, such as *Voyage into the Golden Screen* (Nørgård 1968) and *Symphony No. 2* (Nørgård 1970). In the latter, 4096 notes of the infinity series appear. For analyses of the former piece, see Hansen and Holten (1971); Rasmussen (1993); and Mehta (2011). For the latter piece, see Bisgaard (1974-1975a); Bisgaard (1974-1975b); Christensen (1996). Below, in Figure 2, we reproduce part of the score of *Voyage into the Golden Screen*, where the flute part performs the first few notes of the infinity series.

Figure 2. Excerpt from *Voyage into the Golden Screen* (from (Kullberg 1996)).

Figure 3 illustrates the first 128 values of Nørgård's sequence.

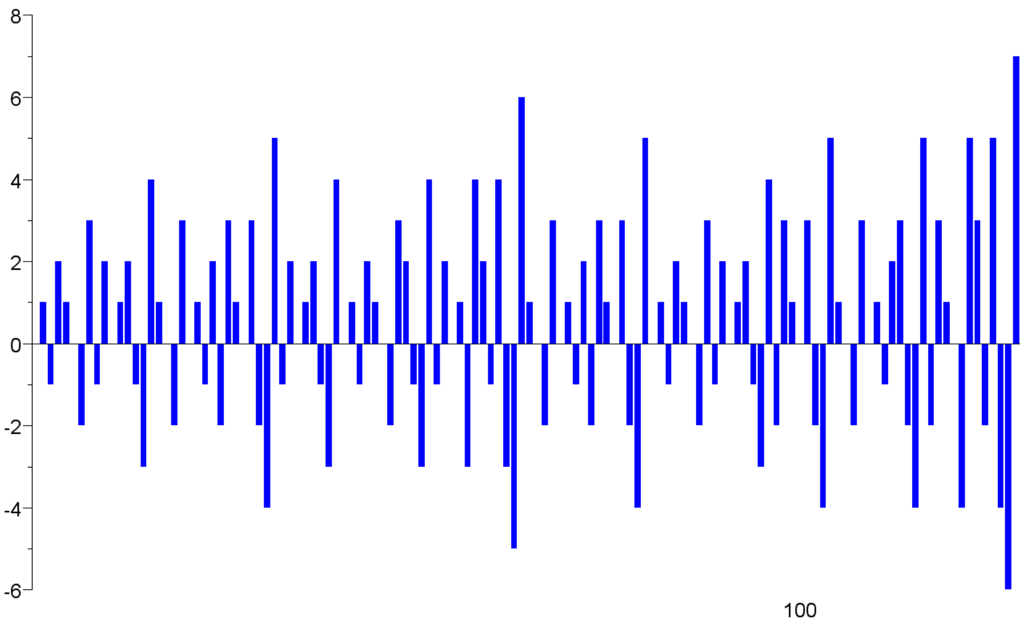

Figure 3. The first 128 values of Nørgård's sequence.

As Figure 3 shows, Nørgård's sequence does not grow very rapidly, a fact that we make mathematically precise in Corollary 4.2.

Although the infinity series has received some study in the music literature (e.g., Christensen (1996); Kullberg (1996); Mehta (2011)), it turns out to have a rich structure that has received comparatively little attention from professional mathematicians. The single exception seems to be the work of Bak (2002). We mention one of Bak's more interesting results: if we define $\eta(X) = \sum_{i \geq 0} s(i) X^i$, the Nørgård power series, then

$$\eta(X) = (X-1)\eta(X^2) + \frac{X}{1-X^2}.$$

Most of the analysis of Nørgård's work has appeared, not surprisingly, in Danish publications: Mortensen (1992) was one of the first to study the sequence mathematically. Mortensen also created a series of web pages Mortensen (2003c,a,b,d, 2005), in both Danish and English, about Nørgård's work.

Unfortunately, not all the claims about Nørgård's sequence in the literature are correct. For example, speaking about Nørgård's sequence modulo 2, Moore (1986) says,

> The numerical infinity series employed by Noergaard does not seem to be related to the types of convergent and divergent infinite series known in mathematics.

In fact, Nørgård's sequence modulo 2 is called the *Thue-Morse sequence*, and has been studied explicitly since 1912 (e.g., Thue (1912); Berstel (1995)).

In this article we examine some novel mathematical aspects of the sequence, and interpret them musically. One of our goals is to put the analysis of Nørgård's infinity series on a more rigorous mathematical footing.

2. Notation and observations

We now fix some notation used throughout the paper. By $(n)_2$ we mean the binary string, having no leading zeros, representing n in base 2. Thus, for example, $(43)_2 = 101011$. Note that $(0)_2$ is the empty string ϵ. If w is a binary string, possibly with leading zeros, then by $[w]_2$ we mean the integer represented by w. Thus, for example, $[0101]_2 = 5$.

By a *block* we mean a finite list of consecutive terms of the sequence. When we interpret the sequence musically, we call this a *phrase*. The block of length j beginning at position i of Nørgård's sequence is denoted by $\mathbf{s}[i..i+j-1]$. If x is a block, then by $|x|$ we mean the length of, or number of notes in, the block x. By x^i, for an integer i, we mean the block of length $i|x|$ consisting of x repeated i times. We write $\operatorname{Card} S$ for the cardinality of a set S.

We recall two basic facts about Nørgård's sequence, both of which follow immediately from the defining recurrence.

Observation 2.1 *If a number a occurs at an even position $n = 2k$, then $1-a$ occurs at position $n = 2k+1$. If a number b occurs at an odd position $n = 2k+1$, then $1-b$ occurs at position $n = 2k$.*

Observation 2.2 *Nørgård's sequence is the fixed point of the map g that sends each integer a to the pair $(-a, a+1)$. Here g is extended in the usual way to finite lists and infinite sequences.*

3. Evaluating Nørgård's sequence

It is useful to have a formula to compute $s(n)$ directly from $(n)_2$, the base-2 expansion of n. The result below can be compared with an essentially equivalent formulation by

Mortensen (2003a).

LEMMA 3.1 *Let* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *be integers with* $a_1, a_2, \ldots, a_n \geq 0$ *and* $b_1, b_2, \ldots, b_n \geq 1$. *If* $w = 1^{b_1}0^{a_1} \cdots 1^{b_n}0^{a_n}$ *then*

$$s([w]_2) = \sum_{1 \leq j \leq n} (-1)^{a_j + \cdots + a_n} b_j.$$

Proof. We proceed by induction on n. The base case is $n = 1$. In this case $w = 1^{b_1}0^{a_1}$, so $[w]_2 = (2^{b_1} - 1)2^{a_1}$. Then $s([w]_2) = s((2^{b_1} - 1)2^{a_1}) = (-1)^{a_1} s(2^{b_1} - 1) = (-1)^{a_1} b_1$, as desired.

For the induction step, assume the result is true for n. Consider $w' = 1^{b_1}0^{a_1} \cdots 1^{b_n}0^{a_n}1^{b_{n+1}}0^{a_{n+1}}$. Then, applying the rules of the recursion, we have $s([w']_2) = (-1)^{a_{n+1}} s([w''_2])$, where $w'' = 1^{b_1}0^{a_1} \cdots 1^{b_n}0^{a_n}1^{b_{n+1}}$. Again applying the rules, we have $s([w'']_2) = b_{n+1} + s([w]_2)$. Putting this all together, and using induction, we get

$$s([w']_2) = (-1)^{a_{n+1}}(b_{n+1} + s([w_2]))$$

or

$$s([w']_2) = \sum_{1 \leq j \leq n+1} (-1)^{a_j + \cdots + a_{n+1}} b_j,$$

as desired. ■

4. The growth rate of Nørgård's sequence

It is natural to ask, when is the first occurrence of a note k in the infinity series? As we will see, the first occurrence is exponentially far out in $|k|$.

PROPOSITION 4.1

(a) If $k \geq 0$, *the least* n *such that* $s(n) = k$ *is* $n = 2^k - 1$;
(b) If $k \leq 0$, *the least* n *such that* $s(n) = k$ *is* $n = 2^{1-k} - 2$.

Proof. A very easy induction, which we omit, shows that $s(2^k - 1) = k$ for all $k \geq 0$. From this it immediately follows that $s(2^{k+1} - 2) = -k$ for all $k \geq 0$.

It remains to prove the following two claims simultaneously by induction on $|k|$:

(a') If $k \geq 0$, then $s(n) = k$ implies $n \geq 2^k - 1$;
(b') If $k \leq 0$, then $s(n) = k$ implies $n \geq 2^{1-k} - 2$.

The reader can verify both claims for $k = 0$. Now assume both (a') and (b') hold for $0 < |k| < m$; we prove them for $k = m$. Suppose $s(m) = k$ and m is the least such index.

We first prove (a'). If m is even then $s(m/2) = -k < 0$. But then (b') implies that $m/2 \geq 2^{k+1} - 2$. So $m \geq 2^{k+2} - 4 \geq 2^k - 1$. If m is odd then $s((m-1)/2) = s(m) - 1 = k - 1 \geq 0$. Then (a') implies that $(m-1)/2 \geq 2^{k-1} - 1$, so $m \geq 2^k - 1$, as desired.

Next, we prove (b'). If m is odd then $s((m-1)/2) = s(m) - 1 = k - 1$. Then $s(m - 1) = -s((m-1)/2) = 1 - k \geq 1$. So by (a') we have $m - 1 \geq 2^{1-k} - 1$. But then

$m \geq 2^{1-k} \geq 2^{1-k} - 2$. If m is even then $s(m/2) = -k$. Since $k < 0$ we have $-k > 0$, and so $m/2 \geq 2^{-k} - 1$ by (a'). Hence $m \geq 2^{1-k} - 2$, as desired. ∎

As a corollary we get tight upper and lower bounds on the value of $s(n)$.

Corollary 4.2 *We have*

$$1 - \log_2(n+2) \leq s(n) \leq \log_2(n+1)$$

for all integers $n \geq 0$, and both bounds are tight infinitely often.

Proof. The bounds clearly hold for $n = 0, 1$. Otherwise write $2^{r-1} < n \leq 2^r$ for some $r \geq 1$, so $r-1 < \log_2(n) \leq r$. From Proposition 4.1 we have $s(n) \leq r-1$ unless $n = 2^r - 1$. Hence, if $n \neq 2^r - 1$ we have $s(n) \leq r - 1 \leq \log_2(n) \leq \log_2(n+1)$. On the other hand, if $n = 2^r - 1$, then $s(n) = \log_2(n+1)$. This proves the upper bound.

For the lower bound, from Proposition 4.1 we have $s(n) \geq 2 - r$ unless $n = 2^r - 2$. Hence, if $n \neq 2^r - 2$ we have $s(n) \geq 2 - r \geq 1 - \log_2(n) \geq 1 - \log_2(n+2)$. On the other hand, if $n = 2^r - 2$, then $s(n) = 1 - \log_2(n+2)$. This proves the lower bound. ∎

5. Note positions

In the previous section we were concerned with the first appearance of a given note in the infinity series. Now we turn to understanding the positions of all such notes.

Given any note a, we let $s^{-1}(a)$ denote the set of all natural numbers n such that $s(n) = a$. For example, for $a = 0$, we have

$$s^{-1}(0) = \{0, 5, 10, 17, 20, 27, 34, 40, 45, 54, 65, 68, 75, 80, 85, 90, 99, 105, 108, \ldots\}.$$

It is then natural to wonder about the complexity of specifying these note positions.

The American linguist Noam Chomsky (1956) invented a famous hierarchy of distinctions among formal languages. The two lowest levels of this hierarchy are the regular languages (those accepted by a finite-state machine) and the context-free languages (those accepted by a finite-state machine with an auxiliary pushdown stack). Here we show that the language of binary representations of $s^{-1}(a)$ is context-free, but not regular.

Theorem 5.1 *For each integer a, the language $L_a = (s^{-1}(a))_2$ is context-free but not regular.*

Proof. It suffices to explain how L_a can be accepted by a pushdown automaton M_a. We assume the reader is familiar with the basic notation and terminology as contained, for example, in Hopcroft and Ullman (1979).

The first part of the construction is the same for all a.

We will design M_a such that, on input n in base 2 (starting from the most significant digit), M_a ends up with $|s(n)|$ counters on its stack, with the sign $m := \text{sgn}(s(n))$ stored in the state. We also assume there is an initial stack symbol Z.

To do this, we use the recursion $s(2n) = -s(n)$ and $s(2n+1) = s(n) + 1$. As we read the bits of n,

- if the next digit read is 0, set $m := -m$;
- if the next digit read is 1, and m is 0 or $+1$, push a counter on the stack, and set $m := +1$;

- if the next digit read is 1, and $m = -1$, pop a counter from the stack and change m to 0 if Z is now on the top of the stack.

The rest of M_a depends on a. From each state where $m = \text{sgn}(a)$, we allow an ϵ-transition to a state that attempts to pop off $|a|$ counters from the stack and accepts if and only if this succeeds, the stored sign is correct, and Z is on top of the stack. This completes the sketch of our construction, and proves that L_a is context-free.

Next, we prove that L_a is not regular. Again, we assume the reader is familiar with the Pumping Lemma for regular languages, as described in Hopcroft and Ullman (1979). Consider $L := L_a \cap 1^*01^*$. From Lemma 3.1, we know that if $n = [1^b01^c]_2$, then $s(n) = c - b$. It follows that

$$L = \{1^i01^{i+a} \ : \ i \geq 0 \text{ and } i + a \geq 0\}.$$

Let n be the Pumping Lemma constant and set $N := n + |a|$. Choose $z = 1^N01^{N+a}$. Then $|z| \geq n$. Suppose $z = uvw$ with $|uv| \leq n$ and $|v| \geq 1$. Then $uw = 1^{N-|v|}01^{N+a} \notin L_a$, since $|v| \geq 1$. This contradiction proves that L_a is not regular. ∎

Interpreted musically, one might say that the positions of every individual note in the infinity series are determined by a relatively simple program (specified by a pushdown automaton), but *not* by the very simplest kind of program. There are indeed computable regularities in these positions, but not finite-state regularities.

6. Counting occurrences of notes

Hansen and Holten (1971) briefly described the distribution of notes in the first 128 notes of the infinity series. However, they apparently did not notice that there is a close relationship between the distribution of the first 2^N notes of the infinity series and the binomial coefficients in Pascal's triangle. For example, for $N = 7$ the number of occurrences of each note is given in Table 3; compare it to row 6 of Pascal's triangle in Table 2.

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Table 2. Rows 0 through 6 of Pascal's triangle.

a	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$r_a(7)$	1	1	6	6	15	15	20	20	15	15	6	6	1	1

Table 3. Distribution of the first 128 notes of Nørgård's sequence.

Here $r_a(N) = \text{Card}\{i : 0 \leq i < 2^N \text{ and } s(i) = a\}$, the number of occurrences of the note a in the first 2^N positions of Nørgård's sequence. The following theorem makes the relationship precise, in terms of the binomial coefficients

$$\binom{n}{t} = \frac{n!}{t!(n-t)!}.$$

THEOREM 6.1 *For all integers a and all $N \geq 1$ we have*

$$r_a(N) = \binom{N-1}{\lfloor (N-a)/2 \rfloor}.$$

Proof. We proceed by induction on N. The base case is $N = 1$, whence $N - 1 = 0$. Then

$$\binom{0}{\lfloor (N-a)/2 \rfloor} = \begin{cases} 1, & \text{if } \lfloor (1-a)/2 \rfloor = 0; \\ 0, & \text{otherwise.} \end{cases}$$

This is 1 if $a \in \{0, 1\}$, and 0 otherwise. But the first 2 notes of $\mathbf{s}$ are 0 and 1, so the result holds.

Now assume the claim holds for $N' < N$; we prove it for N. Now a value of a in $\mathbf{s}[0..2^N - 1]$ can occur in either an even or odd position. If it occurs in an even position, then it arises from $-a$ occurring in $\mathbf{s}[0..2^{N-1} - 1]$. If it occurs in an odd position, then it arises from $a - 1$ occurring in $\mathbf{s}[0..2^{N-1} - 1]$. It follows that, for $N \geq 2$, that

$$r_a(N) = r_{-a}(N-1) + r_{a-1}(N-1).$$

Hence, using induction and the classical binomial coefficient identities

$$\binom{M}{i} = \binom{M-1}{i} + \binom{M-1}{i-1},$$

and

$$\binom{M}{i} = \binom{M}{M-i},$$

we have

$$
\begin{aligned}
r_a(N) &= r_{-a}(N-1) + r_{a-1}(N-1) \\
&= \binom{N-2}{\lfloor (N-1+a)/2 \rfloor} + \binom{N-2}{\lfloor (N-a)/2 \rfloor} \\
&= \begin{cases} \binom{N-2}{(N+a)/2-1} + \binom{N-2}{(N-a)/2}, & \text{if } N \equiv a \pmod 2; \\ \binom{N-2}{(N+a-1)/2} + \binom{N-2}{(N-a-1)/2}, & \text{if } N \not\equiv a \pmod 2; \end{cases} \\
&= \begin{cases} \binom{N-2}{(N-2)-(N+a)/2+1} + \binom{N-2}{(N-a)/2}, & \text{if } N \equiv a \pmod 2; \\ \binom{N-2}{(N-2)-(N+a-1)/2} + \binom{N-2}{(N-a-1)/2}, & \text{if } N \not\equiv a \pmod 2; \end{cases} \\
&= \begin{cases} \binom{N-2}{(N-a)/2-1} + \binom{N-2}{(N-a)/2}, & \text{if } N \equiv a \pmod 2; \\ \binom{N-2}{(N-a-1)/2-1} + \binom{N-2}{(N-a-1)/2}, & \text{if } N \not\equiv a \pmod 2; \end{cases} \\
&= \begin{cases} \binom{N-1}{(N-a)/2}, & \text{if } N \equiv a \pmod 2; \\ \binom{N-1}{(N-a-1)/2}, & \text{if } N \not\equiv a \pmod 2; \end{cases} \\
&= \binom{N-1}{\lfloor (N-a)/2 \rfloor},
\end{aligned}
$$

as desired. ■

Corollary 6.2 *Each note occurs in the infinity series with frequency* 0*, in the limit.*

Proof. This follows immediately from Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}(1+o(1))$. ■

The binomial coefficient appearing in the formula in Theorem 6.1 suggests that there should be a combinatorial bijection between the occurrences of the note a in the first 2^N positions of Nørgård's sequence, and the choice of $\lfloor (N-a)/2 \rfloor$ items from a universe of $N-1$ items. However, the simple proof we gave does not provide such a bijection. Hence, we now reprove Theorem 6.1 in another manner.

Given a positive integer N and another integer n with $0 \le n < 2^N$, we define the word $a_{N,n}$ recursively as follows: let $a_{N,0} := (01)^{N/2}$, and for every $n \ge 1$, let $r = \lfloor \log_2 n \rfloor$. Define

$$a_{N,n} := a_{N,n-2^r} \oplus 0^r 1^{N-r},$$

where $\oplus$ represents addition modulo 2. Thus $a_{N,n}$ is a binary word of length N for all n, and $a_{N,n}$ is obtained by flipping the last $N-r$ bits in $a_{N,n-2^r}$.

The following equivalent, non-recursive definition of $a_{N,n}$ will also be useful. Let $B(n)$ be the exponents of the powers of 2 appearing in the binary representation of n, that is, such that $\sum_{i \in B(n)} 2^i = n$. Thus, for example, $B(25) = \{0, 3, 4\}$. Then it is easy to check that

$$a_{N,n} = (01)^{N/2} \oplus \bigoplus_{i \in B(n)} 0^i 1^{N-i}. \tag{2}$$

For example, the following table lists $a_{4,n}$ for all $0 \leq n < 2^4$, together with the corresponding values of Nørgård's sequence $s(n)$:

n	$s(n)$	$a_{4,n}$
0	0	0101
1	1	1010
2	-1	0010
3	2	1101
4	1	0110
5	0	1001
6	-2	0001
7	3	1110
8	-1	0100
9	2	1011
10	0	0011
11	1	1100
12	2	0111
13	-1	1000
14	-3	0000
15	4	1111

Lemma 6.3 *For every $N \geq 1$, the mapping $f(n) = a_{N,n}$ is a bijection between $\{0, 1, \ldots, 2^N - 1\}$ and the binary strings of length N.*

Proof. It suffices to show that $f(m) \neq f(n)$ whenever $m \neq n$. We prove our claim by induction on N. For $N = 1$, we have $a_{1,0} = 0, a_{1,1} = 1$, so the base case holds. Now for the inductive step, suppose that there do exist $0 \leq m, n < 2^N$ such that $a_{N,m} = a_{N,n}$. Define $m' := m$ if $m < 2^{N-1}$, and $m' := m - 2^{N-1}$ otherwise. By the definition of $a_{N,m}$, we know that $a_{N,m}[1..N-1] = a_{N-1,m'}$. By the same rationale, we see that $a_{N,n}[1..N-1] = a_{N-1,n'}$ if we define n' similarly. Thus, we obtain that $a_{N-1,m'} = a_{N-1,n'}$, which contradicts our inductive hypothesis. ∎

Next, given a binary word x of length N, we define

$$G(x) := 1 + 2\left(\sum_{1 \leq i \leq N} x[i]\right) - N - x[N].$$

Lemma 6.4 *For every $N \geq 1$, and for every n with $0 \leq n < 2^N$, we have $s(n) = G(a_{N,n})$.*

Proof. Fix N. First, it is clear that the claim is true for $n = 0$ and $n = 1$. Let $r = \lfloor \log_2 n \rfloor$. We prove our claim by induction on r. Thus, for the inductive step, we assume that $s(n) = G(a_{N,n})$ whenever $0 \leq n < 2^r$, and prove the equality for $2^r \leq n < 2^{r+1}$.

For convenience, let n' denote $n - 2^r$. We apply the original (recursive) definition of the Nørgård sequence and write $s(n')$ as $\pi_1(\pi_2(\cdots(\pi_k(s(0)))))$, where each

π_i is either the "negate" or "add one" operation. Then it is not hard to see that $s(n) = \pi_1(\pi_2(\cdots(\pi_k(s(1)))))$, for the same operations $\pi_1, \ldots, \pi_k$.

Note that, among the k operations, exactly $|B(n')|$ of them are "add one", and $r - |B(n')|$ are "negate". Thus, we see that $s(n) = s(n') + 1$ if and only if $r - |B(n')|$ is even, and $s(n) = s(n') - 1$ otherwise.

Now, consider $a_{N,n'}$. From equation (2), we see that $a_{N,n'}[r+1..N]$ is either $(01)^{(N-r)/2}$ or $(10)^{(N-r)/2}$. If it is the former, then we know that $a_{N,n} = a_{N,n'}[1..r](10)^{(N-r)/2}$, and so $G(a_{N,n}) = G(a_{N,n'}) + 1$. Otherwise, we have $G(a_{N,n}) = G(a_{N,n'}) - 1$. Now from equation (2) again, $a_{N,n'}[r+1..N] = (01)^{(N-r)/2}$ if and only if $r + B(n')$ is even.

Since $r - B(n')$ and $r + B(n')$ obviously have the same parity, we have shown that $s(n) - s(n') = G(a_{N,n}) - G(a_{N,n'})$ for all n with $2^r \leq n < 2^{r+1}$. This finishes our proof. ■

We are now ready to prove the following corollary.

Corollary 6.5 *For all integers a and integers $N \geq 1$,*

$$\text{Card}\left\{n : 0 \leq n \leq 2^N - 1, s(n) = a\right\} = \binom{N-1}{\lfloor (N+a-1)/2 \rfloor} = \binom{N-1}{\lfloor (N-a)/2 \rfloor}.$$

Proof. From Lemma 6.4, we see that $s(n) = a$ if and only if the first $N - 1$ bits of $a_{N,n}$ have $\lfloor (N+a-1)/2 \rfloor$ ones, and $a_{N,n}[N] \equiv N + a - 1 \pmod 2$. Then we know from Lemma 6.3 that there is a bijection between such strings and occurrences of a in the first 2^N terms of the Nørgård sequence. Thus, the first equality follows. The second follows because, as is well known, $\binom{m}{t} = \binom{m}{m-t}$, and

$$\left\lfloor \frac{N+a-1}{2} \right\rfloor + \left\lfloor \frac{N-a}{2} \right\rfloor = N - 1$$

for all integers a. ■

We note that Mortensen (1992) gave a table with the distribution of notes for the variation on Nørgård's infinity series discussed in Section 11, but apparently did not examine Nørgård's infinity series itself.

7. The running sum of Nørgård's sequence

Define $h(n) = \sum_{0 \leq i \leq n} s(i)$, the running sum of Nørgård's sequence. The following table gives the first few terms of this sequence, which is sequence A274604 in Sloane's *Encyclopedia* (Sloane 2016).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$h(n)$	0	1	0	2	3	3	1	4	3	5	5	6	8	7	4	8	9	9	7	10

Table 4. First few terms of the running sum of Nørgård's sequence.

The quantity $h(n)/(n+1)$ represents the average value of the first $n + 1$ notes of the infinity series. As we will see, this average value is almost exactly 1/2.

Proposition 7.1 *If n is even, say $n = 2k$, then $h(n) = k - s(k)$. If n is odd, say $n = 2k + 1$, then $h(n) = k + 1$.*

Proof. We proceed by induction on n. The reader can check the claim for $n = 0, 1$. Now assume the claim is true for all $n < N$; we prove it for N. Suppose N is even, say $N = 2k$. Then

$$h(N) = h(2k) = s(2k) + h(2k - 1) = -s(k) + k,$$

where we have used the defining equation (1) and induction. On the other hand, if N is odd, say $N = 2k + 1$, then

$$h(N) = h(2k + 1) = s(2k + 1) + h(2k) = (s(k) + 1) + (k - s(k)) = k + 1,$$

where we have used equation (1) and induction. ∎

Similar remarks were made by Bak (2002).

8. Note pairs in the infinity series

We now discuss those pairs (i, j) that occur as two consecutive notes in the infinity series; we call this two-note phrase a *note pair*. If there exists n such that $s(n) = i$ and $s(n + 1) = j$, we say that the note pair (i, j) is *attainable*; otherwise we say it is *unattainable*.

THEOREM 8.1 *The pair (i, j) is attainable if any one of the following conditions hold:*

(a) $i > 0$ and $-i \leq j \leq i - 1$;
(b) $j \geq 1$ and $1 - j \leq i \leq j + 1$ and $i \not\equiv j \pmod 2$;
(c) $j \leq -2$ and $j + 2 \leq i \leq -j - 2$ and $i \equiv j \pmod 2$.

Otherwise (i, j) is unattainable.

Proof. The proof has two parts. In the first part we show that if integers i, j obey any of the conditions (a)–(c) above, then the pair (i, j) is attainable. In the second part of the proof, we show that the remaining pairs are unattainable.

The following cases cover all three cases (a)–(c):

Case 1: $j \leq \min(-1, i - 1)$ and $i \equiv j \pmod 2$. Take $a = -(j + 1)$ and $c = i - (j + 1)$. Then the inequalities imply $a, c \geq 0$ and the congruence implies that c is odd. Take $(n)_2 = 1^a01^c$. Then $(n + 1)_2 = 1^{a+1}0^c$. So $s(n) = c - a = i$ and $s(n + 1) = -(a + 1) = j$.

Case 2: $j \geq \max(2, 1 - i)$ and $i \not\equiv j \pmod 2$. Take $a = j - 1$ and $c = i + j - 1$. Then the inequalities imply $a, c \geq 0$ and the congruence implies that c is even. Take $(n)_2 = 1^a01^c$. Then $(n + 1)_2 = 1^{a+1}0^c$. So $s(n) = c - a = i$ and $s(n + 1) = a + 1 = j$.

Case 3: $i \geq 0$ and $1 - i \leq j \leq 1$ and $i \not\equiv j \pmod 2$. Take $a = 1 - j$ and $c = i + j - 1$. Then the inequalities imply $a, c \geq 0$ and the congruence implies that c is even. Take $(n)_2 = 1^a001^c$. Then $(n + 1)_2 = 1^a010^c$. So $s(n) = a + c = i$ and $s(n + 1) = 1 - a = j$.

Case 4: $i \geq 0$ and $0 \leq j \leq i - 1$ and $i \equiv j \pmod 2$. Take $a = j + 1$ and $c = i - (j + 1)$. Then the inequalities imply $a, c \geq 0$ and the congruence implies that c is odd. Take $(n)_2 = 1^a001^c$. Then $(n + 1)_2 = 1^a010^c$. So $s(n) = a + c = i$ and $s(n + 1) = a - 1 = j$.

Cases (1)–(4) correspond in a somewhat complicated way to parts (a)–(c) of the theorem above. Table 5 below illustrates this correspondence.

$i \backslash j$	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11
−10	1	A	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	A	2
−9	6	1	A	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	A	2	6
−8	1	6	1	A	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	A	2	6	2
−7	6	1	6	1	A	5	5	5	5	5	5	5	5	5	5	5	5	5	5	A	2	6	2	6
−6	1	6	1	6	1	A	5	5	5	5	5	5	5	5	5	5	5	5	A	2	6	2	6	2
−5	6	1	6	1	6	1	A	5	5	5	5	5	5	5	5	5	5	A	2	6	2	6	2	6
−4	1	6	1	6	1	6	1	A	5	5	5	5	5	5	5	5	A	2	6	2	6	2	6	2
−3	6	1	6	1	6	1	6	1	A	5	5	5	5	5	5	A	2	6	2	6	2	6	2	6
−2	1	6	1	6	1	6	1	6	1	A	5	5	5	5	A	2	6	2	6	2	6	2	6	2
−1	6	1	6	1	6	1	6	1	6	1	A	5	5	A	2	6	2	6	2	6	2	6	2	6
0	1	6	1	6	1	6	1	6	1	6	1	A	A	3	6	2	6	2	6	2	6	2	6	2
1	7	1	7	1	7	1	7	1	7	1	7	1	3	7	2	7	2	7	2	7	2	7	2	7
2	1	7	1	7	1	7	1	7	1	7	1	3	4	3	7	2	7	2	7	2	7	2	7	2
3	7	1	7	1	7	1	7	1	7	1	3	1	3	4	2	7	2	7	2	7	2	7	2	7
4	1	7	1	7	1	7	1	7	1	3	1	3	4	3	4	2	7	2	7	2	7	2	7	2
5	7	1	7	1	7	1	7	1	3	1	3	1	3	4	2	4	2	7	2	7	2	7	2	7
6	1	7	1	7	1	7	1	3	1	3	1	3	4	3	4	2	4	2	7	2	7	2	7	2
7	7	1	7	1	7	1	3	1	3	1	3	1	3	4	2	4	2	4	2	7	2	7	2	7
8	1	7	1	7	1	3	1	3	1	3	1	3	4	3	4	2	4	2	4	2	7	2	7	2
9	7	1	7	1	3	1	3	1	3	1	3	1	3	4	2	4	2	4	2	4	2	7	2	7
10	1	7	1	3	1	3	1	3	1	3	1	3	4	3	4	2	4	2	4	2	4	2	7	2
11	7	1	3	1	3	1	3	1	3	1	3	1	3	4	2	4	2	4	2	4	2	4	2	7

Table 5. Illustration of the cases in the proof of Theorem 8.1.

The letter A represents the fact that both cases (5) and (6) below hold.

The pairs not covered by conditions (a)-(c) above can be divided into three parts:

Case 5: $i \leq 0$ and $i - 1 \leq j \leq -i$.

Case 6: $i \leq 0$ and ($j \geq -i$ and $i \equiv j \pmod 2$) or ($j \leq i$ and $i \not\equiv j \pmod 2$).

Case 7: $i > 0$ and ($j \leq -(i+1)$ and $i \not\equiv j \pmod 2$) or ($j \geq i$ and $i \equiv j \pmod 2$).

We need to show all of these pairs are unattainable.

First, we need a lemma.

Lemma 8.2 *Suppose $n = 4k + a$ for $0 \leq a \leq 3$, and $s(n) = i$ and $s(n+1) = j$. Then the values of $s(k)$, $s(2k)$, $s(2k+1)$, and $s(2k+2)$ are as follows.*

a	$s(k)$	$s(2k)$	$s(2k+1)$	$s(2k+2)$	j
0	i	$-i$	$i+1$	—	$1-i$
1	$1-i$	$i-1$	$2-i$	—	$i-2$
2	$-i-1$	$i+1$	$-i$	—	$1-i$
3	$i-2$	$2-i$	$i-1$	$-j$	—

Proof. This follows immediately from the defining recursion. ■

We now show that Case 5 above cannot occur. Choose the smallest possible n such that $s(n) = i$ and $s(n+1) = j$, over all i, j satisfying the conditions $i \leq 0$ and $i - 1 \leq j \leq -i$.

From the table in Lemma 8.2 above we see that if $n = 4k$ or $n = 4k + 2$ we have $j = 1 - i > -i$, a contradiction. Similarly, if $n = 4k + 1$ then $j = i - 2 < i - 1$, a contradiction. Hence $n = 4k + 3$.

Now consider $n' := 2k + 1 = (n-1)/2 < n$. Let $i' := i - 1$ and $j' = -j$. Note that $i' < i \leq 0$ and $i' - 1 = i - 2 < -j - 2 < -j = j'$. However $s(n') = i - 1 = i'$ and $s(n'+1) = s(2k+2) = -s(4k+4) = -j = j'$, contradicting the minimality of n.

Next we show that Case 6 above cannot occur. Choose the smallest possible n such that $s(n) = i$ and $s(n+1) = j$, over all i, j satisfying ($j \geq -i$ and $i \equiv j \pmod 2$) or ($j \leq i$ and $i \not\equiv j \pmod 2$).

From the table in Lemma 8.2 above we see that if $n = 4k$ or $n = 4k+2$ then $j = 1-i > 0$ (since $i \leq 0$). So $i \equiv j \pmod 2$. This contradicts $j = 1 - i$. Similarly, if $n = 4k + 1$, then $j = i - 2$. Since $i \leq 0$ we have $j < 0$ and hence $i \not\equiv j \pmod 2$. This contradicts $j = i - 2$. Hence $n = 4k + 3$.

Now consider $n' := 2k + 1 = (n - 1)/2 < n$. Let $i' := i - 1$ and $j' = -j$. Note that $i' < 0$. There are now two subcases to consider: (i) $j \geq -i$ and $i \equiv j \pmod 2$ and (ii) $j \leq i$ and $i \not\equiv j \pmod 2$.

Subcase (i): The case $j = -i$ is already ruled out by Case 5. So $j \geq 1 - i$. Then $j' = -j \leq i - 1 = i'$. Furthermore $j' \not\equiv i' \pmod 2$. However $s(n') = i - 1 = i'$ and $s(n' + 1) = s(2k + 2) = -s(4k + 4) = -j = j'$, contradicting the minimality of n.

Subcase (ii): The pair where $j = i$ is already unattainable by Case 5. So $j < i$. Then $j' = -j > -i$, implying $j' \geq 1 - i = -i'$. Furthermore $i \equiv j \pmod 2$. Again $s(n') = i - 1 = i'$ $s(n' + 1) = s(2k + 2) = -s(4k + 4) = -j = j'$, contradicting the minimality of n.

Finally, we now show that Case 7 cannot occur. Suppose there is a pair of values $(s(n), s(n + 1)) = (i, j)$ satisfying the conditions $i > 0$ and either $j \leq -(i + 1)$ and $i \not\equiv j \pmod 2$, or $j \geq i$ and $i \equiv j \pmod 2$. Among all such (i, j), let J be the minimum of the absolute values of j. Among all pairs of the form $(i, \pm J)$, let (I, J') be a pair with the smallest value of the first coordinate (which is necessarily positive).

Suppose n is such that $s(n) = I$ and $s(n+1) = J'$. If $n = 4k$ or $n = 4k+2$ then from the table in Lemma 8.2 above we get $J' = 1-I > (-1)-I$ and $J' = 1-I < 0$, a contradiction. If $n = 4k + 1$ then from the table in Lemma 8.2 above we get $J' = I - 2 > -(I + 1)$ (since $I > 0$) and $J' = I - 2 < I$, a contradiction. Hence $n = 4k + 3$.

Then the table in Lemma 8.2 above implies that if we take $n' = 2k+1 = (n-1)/2 < n$, and $I' = I - 1$ and $s(n') = I'$ and $s(n' + 1) = -J'$. If $I' > 0$ then $(I', -J')$ is a pair whose second coordinate has the same absolute value as (I, J), but whose first coordinate is smaller, a contradiction. Otherwise $I' = 0$. But then the pair $(0, -J')$ is not attainable by (b), a contradiction. ■

Next we show that Nørgård's sequence is *recurrent*: every block of notes that occurs in **s**, occurs infinitely many times in **s**.

THEOREM 8.3 *Nørgård's sequence* **s** *is recurrent.*

Proof. Let $(a_0, a_1, \ldots, a_{j-1})$ be a block of j consecutive values of the infinity series, for some $j \geq 1$, that is, suppose there exists n such that $s(n + i) = a_i$ for $0 \leq i < j$. Then there exists a power of 2, say 2^N, such that $j \leq 2^N$. It therefore suffices to show that the block $B := (s(0), s(1), \ldots, s(2^N - 1))$ appears infinitely often.

Consider the block

$$A_t = (s(5 \cdot 2^{N+t}), s(5 \cdot 2^{N+t} + 1), \ldots, s(5 \cdot 2^{N+t} + 2^N - 1)).$$

We claim that $A_t = B$ for all $t \geq 0$. To see this, note that the binary expansion of $5 \cdot 2^{N+t} + i$, for $0 \leq i < 2^N$, looks like 1010^t followed by w, where w is the binary expansion of i padded on the left with zeros to make its length N. It now follows from Lemma 3.1 that $s(5 \cdot 2^{N+t} + i) = a_i$ for $0 \leq i < 2^N$, thus producing a new occurrence of B for each $t \geq 0$. ■

Next we consider the possible intervals[1] that can occur in the infinity series.

[1] "Intervals" here is meant in the musical and not mathematical sense; that is, we mean the number of half

Corollary 8.4 *There exists n such that $s(n+1)-s(n)=k$ if and only if either $k<0$, or $k>0$ and k odd.*

Proof. This follows immediately from Theorem 8.1. ■

Corollary 8.5 *A note pair (a,b) never occurs at both an odd and an even position in* **s**.

Proof. Suppose (a,b) occurs at an even position. Then from the recurrence we have $b=1-a$. If it occurs at an odd position too, say $n=2k+1$, then $s(k)=a-1$ and $s(k+1)=-b$. Then at position k we have the pair $(a-1,a-1)$, which by Theorem 8.1 does not occur. ■

Corollary 8.6 *There are never five or more consecutive non-negative notes in* **s**, *and five is the best possible bound.*

Proof. Assume there five such notes. Then there are four consecutive non-negative notes starting at an even position. From the recursion, these four notes, starting at position n, are of the form $a, 1-a, b, 1-b$. Now $a \geq 0$ and $1-a \geq 0$ imply $a \in \{0,1\}$, and similarly for b. From the recursion, starting at position $n/2$ we must have the notes $(-a,-b)$. However, from Theorem 8.1, none of the note pairs $\{(0,0),(-1,0),(0,-1),(-1,-1)\}$ occur. This contradiction proves the result.

To see that the bound of five is best possible, observe that (2,0,1,2) occurs beginning at $s(9)=2$, and hence by Theorem 8.3 occurs infinitely often in **s**. ■

In a similar fashion, we can prove there are never more than two consecutive positive notes, or two consecutive negative notes, or two consecutive non-positive notes, in the infinity series.

We say a block of notes (a phrase) is *ascending* (respectively, *descending*) if the pitch of each note is greater than (respectively, is less than) that of the immediately preceding note.

Theorem 8.7 *In the infinity series every ascending phrase consists of at most three notes, and every descending phrase consists of at most four notes. Examples of both occur infinitely often.*

Proof. We prove only the claim about ascending phrases, with the descending ones handled similarly.

Assume there exists n such that $s(n) < s(n+1) < s(n+2) < s(n+3)$.

Case 1: n is even. Then $n=2k$. Write $s(k)=a$ and $s(k+1)=b$. Then $s(n)=-a$, $s(n+1)=a+1$, $s(n+2)=-b$, and $s(n+3)=b+1$. Hence $-a<a+1<-b<b+1$, which implies both $a>b$ and $a<b$, a contradiction.

Case 2: n is odd. Then $n=2k+1$. Write $s(k)=a$, $s(k+1)=b$, and $s(k+2)=c$, so that $s(n)=a+1$, $s(n+1)=-b$, $s(n+2)=b+1$, and $s(n+3)=-c$. Hence $a+1<-b<b+1<-c$. If k is even then $b=1-a$, which implies that $a+1<a-1$, a contradiction. If k is odd then $c=1-b$, which implies that $b+1<b-1$, a contradiction.

On the other hand, the phrase $(0,1,2)$ occurs starting at $s(10)=0$, and so infinitely often by Theorem 8.3. ■

steps separating two consecutive notes.

9. Repetitions in the infinity series

Repetitions in sequences have been an object of intense study since the pioneering results of Axel Thue more than a hundred years ago (see Thue (1906, 1912); Berstel (1995)). Thue proved that the Thue-Morse sequence (i.e., the infinity series taken modulo 2) is overlap-free: it contains no block of the form xxa where x is a nonempty block and a is the first number in x.

In this section we characterise close repetitions in the infinity series. We prove that the infinity series has an even stronger avoidance property than the Thue-Morse sequence.

THEOREM 9.1 *If the infinity series contains a block of the form xyx, with x nonempty, then $|y| \geq 2|x|$. In particular $\mathbf{s}$ contains no two consecutive identical blocks.*

Proof. We call a block of notes of the form xyx with $|y| < 2|x|$ a *proximal repetition.* Assume, contrary to what we want to prove, that $\mathbf{s}$ has a proximal repetition xyx occurring for the first time at some position n. Then, without loss of generality, we can assume that $|y|$ is minimal over all proximal repetitions occurring in $\mathbf{s}$. Furthermore, we can assume that n is as small as possible over all occurrences of this xyx in $\mathbf{s}$. Finally, we can assume that $|x|$ is as small as possible over all xyx with $|y|$ minimal occurring at position n. There are a number of cases to consider.

Case 1: $|x| = 1$. If $\mathbf{s}$ contains xyx with $|y| < 2|x|$ then $x = a$ and $y = b$ for single numbers a, b.

If aba occurs beginning at an even position $n = 2k$, then from Observation 2.1, we know that b immediately follows the second a. So $abab$ occurs at position $2k$. Then from the recurrence we know that $(-a)(-a)$ occurs at position k. But from Corollary 8.4 we know that this is impossible.

If aba occurs beginning at an odd position $n = 2k + 1$, then from Observation 2.1, we know that b immediately precedes the first a in $\mathbf{s}$. So $baba$ occurs at position $2k$, and we have already ruled this out in the previous paragraph.

Case 2: $|x| \geq 2$ and $|x| \not\equiv |y| \pmod 2$. Then by considering the first two notes of x, say ab, we have that ab occurs beginning at both an odd and an even position, contradicting Corollary 8.5.

Case 3: $|x| \geq 2$ and both $|x|, |y|$ even. If the block xyx occurs starting at an even position $n = 2k$, then $x'y'x'$ occurs at position k. Now $|x'| = |x|/2$ and $|y'| = |y|/2$ and $x = g(x')$, $y = g(y')$, for g defined in Observation 2.2, so $x'y'x'$ is a proximal repetition occurring at position k. If $|y| > 0$, then $|y'| < |y|$, contradicting our assumption that $|y|$ was minimal. If $|y| = 0$, and $n > 0$, then $x'y'x'$ occurs at position $n/2 < n$, contradicting the assumption that our xyx occurs at the earliest possible position. Finally, if $|y| = 0$ and $n = 0$, then $|x'| < |x|/2$, contradicting the minimality of $|x|$.

Otherwise xyx occurs starting at an odd position $n = 2k + 1$. If $|y| = 0$ then write $x = wa$ for a single number a. Since $|x|$ is even and $xyx = wawa$ occurs beginning at an odd position, the first a is at an even position. So another a immediately precedes the first w. Then $awaw$ occurs starting at position $n = 2k$. This contradicts our assumption that xyx was the earliest occurrence.

Otherwise $|y| > 0$. Write $x = aw$ for a single letter a and $y = zb$ for a single letter b. Note that $|z|$ is odd. Since both $|x|$ and $|y|$ are even, b occurs at an even position and immediately precedes the second occurrence of x. So b also immediately precedes the first occurrence of x. Thus $bxyx = bawzbaw$ occurs at position $2k$. Then $|z| < |y| < 2|x| < 2|baw|$, so $(baw)z(baw)$ is a proximal repetition with $|z| < |y|$. This contradicts our assumption that $|y|$ was minimal.

Case 4: $|x| \geq 2$ and both $|x|, |y|$ odd. Suppose xyx begins at an even position, say $n = 2k$. Then, writing $y = az$ for a single number a, we see that a immediately follows the first x and occurs at an odd position. So a also follows the second x and we know $xazxa$ occurs at position n. Since $|xa|$ and $|z|$ are both even, there exist x', y' with $g(x') = xa$ and $g(y') = z$. So $x'y'x'$ occurs at position k. However $|y'| = |z|/2 = (|y| - 1)/2 < |x| - 1/2 < |x| = 2|x'| - 1$, and so $x'y'x'$ is a proximal repetition with $|y'| < |y|$. This contradicts our assumption that $|y|$ was minimal.

Similarly, if xyx begins at an odd position, say $n = 2k + 1$, then we can write $y = za$ for a single number a. Then a occurs at an even position and immediately precedes the second x, so it also occurs immediately before the first x. Thus $axzax$ occurs at position $n = 2k$. Then we can argue about ax and z exactly as in the preceding paragraph to get a contradiction. ■

We remark that Theorem 9.1 is optimal since, for example, at position 1 of **s** we have $(1, -1, 2, 1)$, which corresponds to $x = 1$, $y = (-1, 2)$ and $|y| = 2|x|$. By applying g to this occurrence we find larger and larger occurrences of xyx satisfying the same equality. For example, by consulting Figure 1, we see that the four-note phrase $(A\flat, G, F\natural, B\flat)$ occurs in measure 2 and measure 5, separated by the 8 notes of measures 3 and 4.

Musically, we may say that although each phrase in the infinity series occurs infinitely often, we never hear exactly the same phrase twice without a relatively long delay between the two occurrences. This may partially account for the impression of neverending novelty in the music.

10. Characterizing the infinity series

The infinity series has been called unique (e.g., Mortensen (2003b)). To make this kind of assertion mathematically rigorous, however, we need to decide on the main properties of the sequence to see if there could be other sequences meeting the criteria.

Although these main properties could be subject to debate, here are a few of the properties of **s** observed by us and others:

1. s is a surjective map from $\mathbb{N}$ to $\mathbb{Z}$. That is, for all $a \in \mathbb{Z}$ there exists n such that $s(n) = a$.

Proof. If $a \geq 0$, then it is easy to see that $s(2^a - 1) = a$. If $a < 0$, then it is easy to see that $s(2^{1-a} - 2) = a$. ■

2. It is k-self-similar (Mortensen 2003b). That is, there exists a $k \geq 2$ such that for all $i \geq 0$ and all j with $0 \leq j < k^i$, the subsequence $(s(k^i n + j))_{n \geq 0}$ is either of the form $s(n) + a$ for some a (in other words, the sequence transposed by a half steps) or of the form $-s(n) + a$ for some a (in other words, the sequence inverted and then transposed by a half steps). In the Nørgård sequence, we have $k = 2$.

3. Every interval occurs. That is, for all $i \neq 0$, there are two consecutive notes that are exactly i half steps apart. More precisely, for all $i > 0$ there exists n such that

$$|s(n + 1) - s(n)| = i.$$

Note, however, that by Theorem 8.1, it is *not* true that every possible note pair occurs somewhere. In fact, asymptotically, only half of all possible note pairs occur. Furthermore, some intervals occur only in a descending form; by Corollary 8.4 this is true exactly of all even intervals.

4. Every interval that occurs, occurs beginning at infinitely many different notes. That is, for all negative i and all positive odd i, there are infinitely many distinct j such that there exists n with $s(n) = j$ and $s(n+1) = i + j$.

5. Runs of consecutive negative (respectively, positive, non-negative, non-positive) notes are of bounded length (Theorem 8.6).

6. Ascending and descending phrases are of bounded length (Theorem 8.7).

7. It is recurrent; that is, to say, every block of values that occurs, occurs infinitely often (Theorem 8.3).

8. It grows slowly, that is, $|s(n)| = O(\log n)$, and furthermore there exists a constant c such that $|s(n)| > c \log n$ infinitely often (Corollary 4.2). Musically, this corresponds to novel notes appearing infinitely often, but with longer and longer delays between their first appearances.

9. It is non-repetitive or "squarefree". That is, for all $n \geq 1$ and $i \geq 0$ the phrase given by the notes $(s(i), s(i+1), \ldots, s(i+n-1))$ is never followed immediately by the same phrase repeated again. We proved an even stronger statement for Nørgård's sequence in Theorem 9.1.

Are there other sequences with the same nine properties? Of course the negated sequence $-\mathbf{s} = (-s(n))_{n\geq 0}$ also satisfies them, but a brief computer search turned up many others. For example, consider the sequence $\mathbf{t} = (t(n))_{n\geq 0}$ given by the rules

$$\begin{aligned} t(0) &= 0 \\ t(4n) &= t(n) \\ t(4n+1) &= t(n) - 2 \\ t(4n+2) &= -t(n) - 1 \\ t(4n+3) &= t(n) + 2. \end{aligned}$$

The first few terms are depicted in Table 6.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$t(n)$	0	−2	−1	2	−2	−4	1	0	−1	−3	0	1	2	0	−3	4	−2	−4	1

Table 6. The sequence $\mathbf{t}$.

Interpreted musically, starting at C, the first 64 notes of this sequence are depicted in Figure 4.

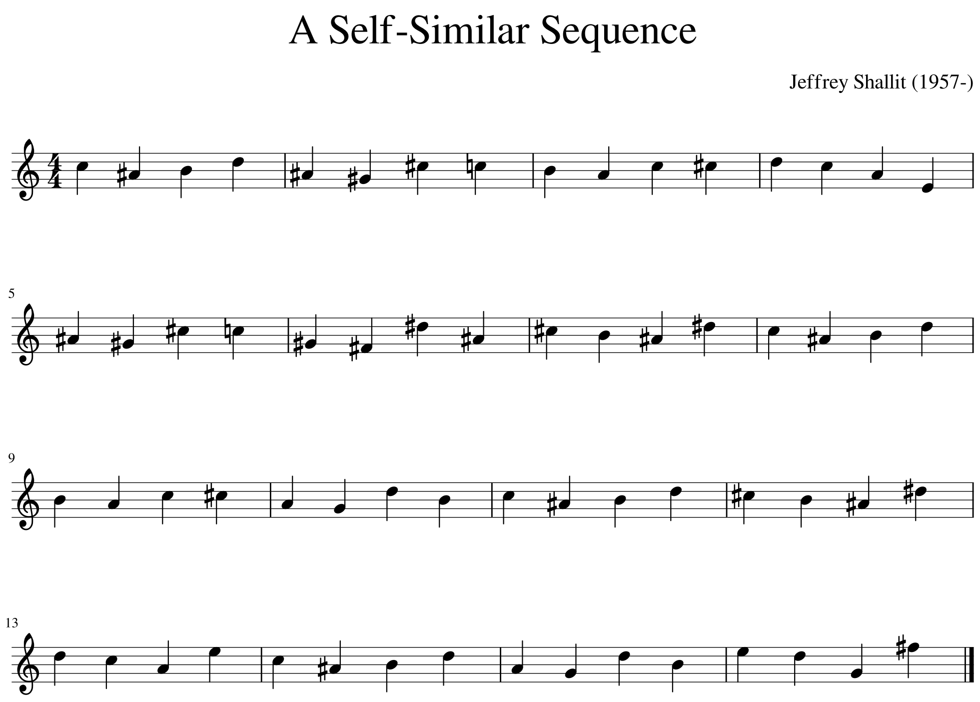

Figure 4. A self-similar sequence consisting of the first 64 notes of **t**, starting with $C = 0$.

It is not hard to prove, along the lines of the the proofs given above, that **t** shares all nine properties with **s**. So in fact, **s** is not really unique at all, at least mathematically speaking.

11. Variations

Nørgård also explored variations on the infinity series. The descriptions given in Mortensen (1992; 2003d; 2005) are a little imprecise, so we reformulate them below.

First variation:

$$\begin{aligned} u(0) &= 0 \\ u(3n) &= -u(n) \\ u(3n+1) &= u(n) - 2 \\ u(3n+2) &= u(n) - 1 \end{aligned}$$

The first few terms are given in Table 7.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$u(n)$	0	−2	−1	2	−4	−3	1	−3	−2	−2	0	1	4	−6	−5	3	−5	−4	−1

Table 7. Mortensen's first variation on the infinity series.

Note that this sequence has a repetition of order three ("cube") beginning at position 32: $(-1, -1, -1)$. This, together with the recursion, ensures that there will be arbitrarily large such repetitions in the sequence. It fails property 9 of the previous section.

Second variation:

$$\begin{aligned} v(0) &= 0 \\ v(3n) &= -v(n) \\ v(3n+1) &= v(n) - 3 \\ v(3n+2) &= -2 - v(n) \end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$v(n)$	0	−3	−2	3	−6	1	2	−5	0	−3	0	−5	6	−9	4	−1	−2	−3	−2

Table 8. Mortensen's second variation on the infinity series.

This sequence fails to have property 3: not every interval occurs. In fact, only intervals of an odd number of half steps occur. However, it has all the other properties listed in the previous section.

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