# On the Polyhedral Lift-and-Project Methods and the Fractional Stable Set Polytope

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May 20, 2008 (revised: October 3, 2008)
submitted to *Discrete Optimization*

**Abstract**

We study two polyhedral lift-and-project operators (originally proposed by Lovász and Schrijver in 1991) applied to the fractional stable set polytopes. First, we provide characterizations of all valid inequalities generated by these operators. Then, we present some seven-node graphs on which the operator enforcing the symmetry of the matrix variable is strictly stronger on the odd-cycle polytope of these graphs than the operator without this symmetry requirement. This disproves a conjecture of Lipták and Tunçel from 2003.

**Keywords:** stable set problem, lift-and-project, integer programming
**AMS Subject Classification:** 90C10, 90C27, 47D20

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# 1 Introduction

The area of discrete optimization has had many fruitful interactions with the areas of linear programming and graph theory. One of the main approaches starts with a 0,1 integer programming formulation of the given discrete optimization problem, focuses on its linear programming relaxation, and works towards better and better approximations of the convex hull of integer solutions of the linear programming relaxation.

Let $P \subseteq [0,1]^n$ be the polytope representing the feasible region of the linear programming relaxation of the given discrete optimization problem. Let us denote the convex hull of integer points in $P$ by $P_I$:

$$P_I := \operatorname{conv}\left(P \cap \{0,1\}^n\right).$$

*Polyhedral lift-and-project methods* start from a description of $P$ for which linear programming is easy or at least tractable (e.g., either $P$ is given by an explicit list of linear inequalities, or a polynomial time separation oracle for $P$ is available) and "generate" a sequence of polytopes converging to $P_I$ in at most $n$ major steps. Usually, each major step (iteration) of a lift-and-project method is described by an operator on the space of polytopes. In this paper, we focus on two polyhedral lift-and-project methods whose operators are denoted by $N_0(\cdot)$ and $N(\cdot)$.

Let $P \subseteq [0,1]^n$ be a polytope. We define a polyhedral convex cone $K(P) \subset \mathbb{R}^{n+1}$ corresponding to $P$,

$$K(P) := \operatorname{cone}\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} : x \in P \right\}.$$

We will refer to the special, homogenizing coordinate as the zeroth coordinate. If $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and $P$ is nonempty, then

$$K(P) = \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1} : Ax \leq x_0 b \right\}.$$

We denote by $e_j$ the $j$th unit vector of suitable size (the size is determined by the context). To approximate $P_I$ better, we can define another polyhedral relaxation of it in the *lifted*, matrix space:

$$\begin{aligned} M_0(P) \quad := \quad & \Big\{ Y \in \mathbb{R}^{(n+1)\times(n+1)} : \operatorname{diag}(Y) = Ye_0 = Y^T e_0, \\ & Ye_j, Y(e_0 - e_j) \in K(P), \forall j \in \{1,2,\ldots,n\} \Big\}. \end{aligned}$$

We *project* back onto the original space of $P$ and obtain our next relaxation:

$$N_0(P) := \left\{ x \in \mathbb{R}^n : \ \exists Y \in M_0(P), Ye_0 = \begin{pmatrix} 1 \\ x \end{pmatrix} \right\}.$$

Note that in the above definition of $N_0$, if we require matrix $Y$ to be symmetric, we still end up with a polytope containing $P_I$ and contained in $N_0(P)$. So, we also define

$$M(P) := \left\{ Y : Y \in M_0(P), Y = Y^T \right\},$$

and

$$N(P) := \left\{ x \in \mathbb{R}^n : \ \exists Y \in M(P), Ye_0 = \begin{pmatrix} 1 \\ x \end{pmatrix} \right\}.$$

These two lift-and-project operators, $N(\cdot)$ and $N_0(\cdot)$, were proposed by Lovász and Schrijver [16]. Independently of [16], Sherali and Adams also proposed and studied similar lift-and-project operators [19] (also see Sherali and Adams [18] for various generalizations and wide applications). There are many related lift-and-project operators; for work in the early 1970's see Balas [4] and Balas, Ceria and Cornuéjols [5]; for a more recently proposed operator, see Bienstock and Zuckerberg [7]. For comparisons among various lift-and-project operators (and in some cases valid inequalities obtained from other procedures) see [8, 9, 10, 12, 13].

One of the main application areas for lift-and-project methods has been the packing/covering type discrete optimization problems (see [1, 2, 6, 11, 17, 20, 21]). Here, we focus on a very well-known problem from this family. We let $\mathrm{STAB}(G)$ denote the *stable set polytope* of $G$, which is the convex hull of the incidence vectors of the stable sets of $G$. Some of the most popular applications of lift-and-project methods in the literature have been based on the polyhedral relaxations of the stable set polytope.

Perhaps the simplest approximation to $\mathrm{STAB}(G)$ obtained from the linear programming relaxation of an integer programming formulation is $\mathrm{FRAC}(G)$, the *fractional stable set polytope* of a graph $G$:

$$\mathrm{FRAC}(G) := \left\{ x \in [0,1]^{V(G)} : x_u + x_v \leq 1, \ \forall \{u,v\} \in E(G) \right\},$$

where $V(G)$, $E(G)$ denote the node set and the edge set of graph $G$ respectively. For every graph $G$, $\mathrm{STAB}(G)$ is precisely the convex hull of integer points in $\mathrm{FRAC}(G)$. In general, $\mathrm{FRAC}(G) \neq \mathrm{STAB}(G)$ unless $G$ is bipartite.

Let $N_0^k(G)$ (resp. $N^k(G)$) denote the polytope we obtain from applying $N_0$ (resp. $N$) successively to $\mathrm{FRAC}(G)$ $k$ times. Lovász and Schrijver [16] proved

$$N_0(G) = N(G) = \mathrm{OC}(G),$$

where $\mathrm{OC}(G)$ denotes the polytope defined by intersecting $\mathrm{FRAC}(G)$ with all the odd-cycle inequalities for $G$. Lipták [14] further analyzed the valid inequalities and facets of $N_0^k(G)$ and $N^k(G)$, for $k \geq 2$. Many of the existing results in the area showed that $N_0^k(G)$ and $N^k(G)$ exhibited similar behaviour (see [16, 14, 15]). Lipták and the second author conjectured that

$N$-$N_0$ *Conjecture* [15]: for every $k \in \mathbb{Z}_+$, $N_0^k(G) = N^k(G)$ for all graphs $G$.

Here, we provide a counter-example to this conjecture.

## 2 Preliminaries

Suppose $P$ is given as

$$P = \{x \in \mathbb{R}^n : Ax \leq b\},$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. We express $N_0(P)$ and $N(P)$ in terms of the valid inequalities derived from the system $Ax \leq b$. $\mathrm{Null}_{\min}(A)$ denotes the minimal elements (with respect to their support) in the null space of $A$. (In other words, this is the set of minimal linear dependencies among the columns of $A$.) We denote by $A_j$ the $j$th column of $A$ and $A \setminus A_j$ denotes the $m$-by-$(n-1)$ matrix obtained from $A$ by removing its $j$th column. We are interested in the minimal elements in the null space of $[A \setminus A_j]^T$ for each $j$:

$$\mathcal{U}_0(A; j) := \left\{ u \in \mathbb{R}^m : u \in \mathrm{Null}_{\min}\left( [A \setminus A_j]^T \right) \right\}.$$

For every $u \in \mathbb{R}^m$, we define $u_-, u_+ \in \mathbb{R}^m$ by

$$(u_-)_i := \max\{0, -u_i\}, \quad (u_+)_i := \max\{0, u_i\}$$

so that $u = u_+ - u_-$.

**Theorem 1.** *Let $P \subseteq [0,1]^n$ be given as above. Then,*

$$N_0(P) = \bigcap_{j=1}^{n} \left\{ x \in P : u^T \left(A_j - b\right) x_j + u_-^T A x \le u_-^T b, \forall u \in \mathcal{U}_0(A;j) \right\}.$$

*Proof.* From the definition of $N_0$, we have $x \in N_0(P)$ if and only if $\exists X \in \mathbb{R}^{n \times n}$ such that $Y := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in M_0(P)$. The conditions $Ye_j, Y(e_0 - e_j) \in K(P)$ are equivalent to $AX_j \le x_j b, A(x - X_j) \le (1 - x_j) b$, where we denoted by $X_j$ the $j$th column of $X$. Since $\operatorname{diag}(X) = x$, we can eliminate the variables $X_{ii}$ and write $N_0(P) = \{x : \exists w, Cx + Dw \le f\}$ where

$$C := \begin{pmatrix} A_1 - b & 0 & \dots & 0 \\ 0 & A_2 - b & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n - b \\ b & A_2 & \dots & A_n \\ A_1 & b & \dots & A_n \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & b \end{pmatrix},$$

$$D := \begin{pmatrix} A \setminus A_1 & 0 & \dots & 0 \\ 0 & A \setminus A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \setminus A_n \\ -(A \setminus A_1) & 0 & \dots & 0 \\ 0 & -(A \setminus A_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -(A \setminus A_n) \end{pmatrix} \quad \text{and} \quad f := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \\ b \\ \vdots \\ b \end{pmatrix}.$$

We are interested in a description of $N_0(P)$ in terms of only $x$, so we define

$$L := \left\{ u \ge 0 : u \in \operatorname{Null}(D^T) \right\},$$

and using the theorem of the alternative:

$$\exists w : Dw \le f - Cx \iff \nexists u \ge 0 : D^T u = 0, u^T (f - Cx) < 0,$$

we have $N_0(P) = \left\{ x : u^T C x \le u^T f, \forall u \in L \right\}$. Furthermore, we only have to consider $u$'s that are extreme rays of $L$, as the inequalities they generate imply those generated by all other $u$'s in $L$. Since $L$ is the intersection of a linear subspace and the non-negative orthant, its extreme rays are exactly the elements which are minimal with respect to their supports.

Now given $u \in \mathbb{R}_+^{2mn}$, let $u^j := [u_{jm+1}, u_{jm+2}, \ldots, u_{(j+1)m}]^T$ for every $j \in \{1, \ldots, 2n\}$. Then we see that $u \in L \iff (u^j - u^{n+j})^T \in \mathrm{Null}([A \setminus A_j]^T)$, $\forall j \in \{1, \ldots, n\}$. By minimality, $\exists j \in \{1, \ldots, n\}$ such that $u^\ell = 0\ \forall \ell \notin \{j, n+j\}$. Moreover, either $u^j = u^{n+j}$ and are both a multiple of some unit vector, or $\exists v \in \mathcal{U}_0(A; j)$ such that $u^j = v_+, u^{n+j} = v_-$.

Finally we consider the inequality $u^T Cx \le u^T f$. In the case when $u^j = u^{n+j}$, we get exactly the inequalities that define $P$. In the other case, we get $v^T(A_j - b)x_j + v_-^T Ax \le v_-^T b$ as claimed. □

To characterize $N(P)$, we define

$$\mathcal{U}(A) := \left\{ U \in \mathbb{R}^{m \times n} : \left(U^T A\right)_{ij} = -\left(U^T A\right)_{ji}, \forall i \neq j \right\}.$$

**Theorem 2.** *Let $P \subseteq [0,1]^n$ be given as above. Then,*

$$N(P) = \left\{ x \in \mathbb{R}^n : \left[ \mathrm{diag}(U^T A) - U^T b + A^T \sum_{j=1}^n (U_j)_- \right]^T x \le b^T \sum_{j=1}^n (U_j)_-, \quad \forall U \in \mathcal{U}(A) \right\}.$$

*Proof.* As in the above proof of Theorem 1, we can find matrices $C', D'$ and a vector $f'$ such that $N(P) = \{x : \exists w, C'x + D'w \le f'\}$. Notice that $C' = C, f' = f$ and

$$D' := \begin{pmatrix} A \setminus A_1 & 0 & 0 & \ldots & 0 \\ A_1 \otimes e_1^T & A \setminus (A_1, A_2) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 \otimes e_{n-2}^T & A_2 \otimes e_{n-3}^T & A_3 \otimes e_{n-4}^T & \ldots & A \setminus (A_1, \ldots A_{n-1}) \\ A_1 \otimes e_{n-1}^T & A_2 \otimes e_{n-2}^T & A_3 \otimes e_{n-3}^T & \ldots & A_{n-1} \otimes e_1^T \\ -(A \setminus A_1) & 0 & 0 & \ldots & 0 \\ -(A_1 \otimes e_1^T) & -(A \setminus (A_1, A_2)) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(A_1 \otimes e_{n-2}^T) & -(A_2 \otimes e_{n-3}^T) & -(A_3 \otimes e_{n-4}^T) & \ldots & -(A \setminus (A_1, \ldots A_{n-1})) \\ -(A_1 \otimes e_{n-1}^T) & -(A_2 \otimes e_{n-2}^T) & -(A_3 \otimes e_{n-3}^T) & \ldots & -(A_{n-1} \otimes e_1^T) \end{pmatrix}.$$

In the above, $\otimes$ denotes the tensor (Kronecker) product and as before, $e_j$ denotes the $j$th unit vector of suitable size (note that the block columns in $D'$ contain $n-1$, $n-2$, …, 1 columns respectively). Also, define $L' := \left\{ u \ge 0 : u \in \mathrm{Null}(D'^T) \right\}$, then $u \in L' \iff (u^i - u^{n+i})^T A_j = -(u^j - u^{n+j}) A_i$ for all distinct $i, j$'s. If we define the matrix $U$ by defining its columns as $U_j := u^j - u^{n+j}\ \forall j \in \{1, \ldots, n\}$, then we have $u \in L' \iff U \in \mathcal{U}(A)$, and the inequality $u^T C' x \le u^T f'$ is exactly

$$\left[ \mathrm{diag}(U^T A) - U^T b + A^T \sum_{j=1}^n (U_j)_- \right]^T x \le b^T \sum_{j=1}^n (U_j)_- .$$

□

The matrix variables $U$ involved in the above theorems provide certificates of the validity of an inequality for $N_0^{k+1}(P)$ as a function of the facets of $N_0^k(P)$ (similarly for $N^{k+1}(\cdot)$). Note that if we are trying to generate a cretain type of valid inequality for $N_0^{k+1}(P)$ (or $N^{k+1}(P)$) obeying some linear equations and inequalities, we can write a linear optimization problem in the $U$ variables and solve it to derive such a valid inequality or to prove that it does not exist. For example, if we want to derive a valid inequality for $N_0^{k+1}(P)$: $a^T x \leq \gamma$ such that $a \geq 0$, $a_i = a_j, \forall i, j \in J_1$ (these constraints may be required due to certain symmetries in our $P$) and $a_i = 2a_k, i \in J_2$ (perhaps such constraints may be motivated by adhoc observations on $P$) for some subsets $J_1, J_2 \subset \{1, 2, \ldots, n\}$, we can express all these conditions as the constraints of a linear optimization problem whose optimal solution (if it exists) would give a certificate of the validity of $a^T x \leq \gamma$. In full generality, we may require that the coefficients of the valid inequality lie in a polyhedron defined by $F \begin{bmatrix} a \\ \gamma \end{bmatrix} = h, a \geq 0$. Consider for instance the following LP problem which tries to compute such $u$'s (and hence the corresponding valid inequality $a^T x \leq \gamma$) for $N_0(P)$:

$$\begin{array}{lrcl} \min & \sum_{i=1}^{m} (u_-)_i + (u_+)_i \,, & & \\ & F \begin{bmatrix} A^T u_- + (u_- + u_+)^T (A_j - b) e_j \\ b^T u_- \end{bmatrix} & = & h, \\ & A^T u_- + (u_- + u_+)^T (A_j - b) e_j & \geq & 0, \\ & u_- \geq 0, \; u_+ \geq 0. & & \end{array}$$

We will soon see examples of such $U$ matrices.

We call a set $P \subseteq \mathbb{R}^n_+$ *lower-comprehensive* if $\forall x \in P$, $0 \leq y \leq x$ implies $y \in P$. Note that for every graph $G$, FRAC($G$), OC($G$) and STAB($G$) are all lower-comprehensive polytopes. Moreover, it is well-known that the operators $N(\cdot)$ and $N_0(\cdot)$ preserve the lower-comprehensiveness of the argument.

For every two dimensional lower-comprehensive polytope $P$, both operators $N$ and $N_0$ generate the same sequence of polytopes converging to $P_I$. Since in this simple case, only two iterations of $N_0$ suffice to reach $P_I$, to conclude that $N_0^k(P) = N^k(P)$ for every $k$, the following simple fact is enough.

**Proposition 3.** *For every lower-comprehensive convex set $P \subseteq [0,1]^2$, $N_0(P) = N(P)$.*

*Proof.* Lovász and Schrijver [16] gave the following geometric characterization of $N_0$:

$$N_0(P) = \bigcap_{j=1}^{n} \operatorname{conv}\{x \in P : x_j \in \{0,1\}\}. \tag{1}$$

Therefore, if $P$ does not contain either $[0,1]^T$ or $[1,0]^T$, then $N_0(P) = P_I$, and so $N_0(P) = N(P)$. Otherwise, we define $\bar{x}_1 := \max\left\{x_1 : [x_1, 1]^T \in P\right\}$ and $\bar{x}_2 := \max\left\{x_2 : [1, x_2]^T \in P\right\}$. If $\bar{x}_1 = \bar{x}_2 = 0$, then again $N_0(P) = P_I$. Otherwise (by equation (1)),

$$N_0(P) = \operatorname{conv}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \frac{1}{\bar{x}_1 + \bar{x}_2 - \bar{x}_1 \bar{x}_2} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \right\}.$$

Obviously, $N(P)$ contains the first three of the above points (since $P_I$ does). For the last one, we see that

$$\begin{pmatrix} \bar{x}_1 + \bar{x}_2 - \bar{x}_1 \bar{x}_2 & \bar{x}_1 & \bar{x}_2 \\ \bar{x}_1 & \bar{x}_1 & \bar{x}_1 \bar{x}_2 \\ \bar{x}_2 & \bar{x}_1 \bar{x}_2 & \bar{x}_2 \end{pmatrix} \in M(P).$$

Hence, $\frac{1}{\bar{x}_1+\bar{x}_2-\bar{x}_1\bar{x}_2}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} \in N(P)$ as well. The convexity of $N(P)$ implies that $N(P) \supseteq N_0(P)$. Since the reverse containment always holds by definition, we deduce $N_0(P) = N(P)$. □

So, a natural first response to $N$-$N_0$ Conjecture would be to wonder whether the property of lower-comprehensiveness is enough to guarantee equality between $N$ and $N_0$. However, this is not so even for three-dimensional, lower-comprehensive polytopes as was pointed out in [15]. Below, we give an example.

**Example 4.** *Let* $P := \left\{x \in [0,1]^3 : 3x_1 + 3x_2 + x_3 \leq 5\right\}$. *Then* $N_0(P) = \left\{x \in [0,1]^3 : \tilde{A}x \leq \tilde{b}\right\}$ *where*

$$\tilde{A} := \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 3 & 0 \\ 3 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{b} := \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix}.$$

*On the other hand,* $N(P) = N_0(P) \cap \{x : 15x_1 + 15x_2 + 5x_3 \leq 23\}$. *The assignment of weights on the inequalities of* $P$ *that induces the latter inequality is (we only list the nonzero elements of* $U_{ij}$*):*

| $i$ | $j$ | $U_{ij}$ |
|---|---|---|
| $3x_1 + 3x_2 + x_3 \leq 5$ | 1 | 3 |
| | 2 | 3 |
| | 3 | $-1$ |
| $x_1 \leq 1$ | 2 | $-18$ |

*yielding*

$$U^T A = \begin{pmatrix} 9 & 9 & 3 \\ -9 & 9 & 3 \\ -3 & -3 & -1 \end{pmatrix}$$

*which satisfies* $(U^T A)_{ij} = -(U^T A)_{ji}\ \forall i \neq j$, *and hence* $U \in \mathcal{U}(A)$. *The only extreme point in* $N_0(P)$ *that violates the inequality* $15x_1 + 15x_2 + 5x_3 \leq 23$ *is* $\frac{1}{4}[3,3,1]^T$.

## 3 Counter-examples to the $N_0^k(G) = N^k(G)$ conjecture

Here we give an example for which $N_0^2(G) \neq N^2(G)$, hence disproving the $N$-$N_0$ Conjecture.

**Claim 5.** *Let* $G_1$ *be the graph in Figure 1. Then*

$$\bar{x} := \frac{1}{5}[2,2,1,2,1,1,1]^T \in N_0^2(G_1) \setminus N^2(G_1).$$

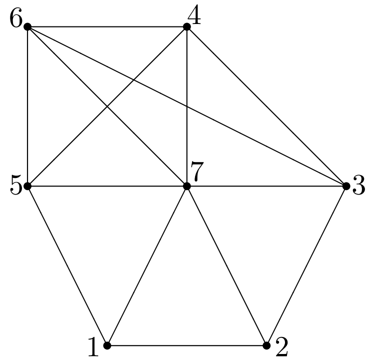


Figure 1: A graph $G_1$ such that $N^2(G_1) \neq N_0^2(G_1)$

*Proof.* To show that $\bar{x} \in N_0^2(G_1)$, we consider the following matrix

$$\frac{1}{5}\begin{pmatrix} 5 & 2 & 2 & 1 & 2 & 1 & 1 & 1 \\ 2 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that every column and the difference of every column with the zeroth column belong to $K(\mathrm{OC}(G))$. Thus, the matrix above is in $M_0^2(G_1)$, and consequently $\bar{x} \in N_0^2(G_1)$.

Next, we give a valid inequality of $N^2(G_1)$ that $\bar{x}$ violates. Consider the following assignment of weights on the inequalities:

| $i$ | $j$ | $U_{ij}$ |
|---|---|---|
| $x_3 + x_4 + x_6 \leq 1$ | 1 | 1 |
| $x_4 + x_5 + x_6 \leq 1$ | 2 | 1 |
| $x_1 + x_2 + x_7 \leq 1$ | 3 | $-1$ |
| | 4 | $-1$ |
| | 5 | $-1$ |
| | 6 | $-1$ |
| $x_2 + x_3 \leq 1$ | 3 | 1 |
| $x_1 + x_5 \leq 1$ | 5 | 1 |
| $x_3 + x_7 \leq 1$ | 3 | 1 |
| $x_4 + x_7 \leq 1$ | 4 | 1 |
| $x_5 + x_7 \leq 1$ | 5 | 1 |
| $x_6 + x_7 \leq 1$ | 6 | 1 |

Then the inequality generated by $U$ is $[3, 3, 1, 1, 1, 1, 4]x \leq 4$, which is not valid for $\bar{x}$. Therefore, our claim follows. ☐

**Remark 6.** *The inequality* $[3, 1, 1, 1, 3, 1, 4]x \leq 4$ *is in fact a facet of* $N^2(G_1)$. *The other facets of* $N^2(G_1)$ *that are not valid for* $N_0^2(G_1)$ *are* $[3, 3, 2, 2, 2, 1, 5]x \leq 5$ *and* $[3, 3, 2, 1, 2, 2, 5]x \leq 5$.

### 3.1 A perfect graph counter-example

In fact, Claim 5 still holds if we add the edge $\{3, 5\}$ to the above graph.

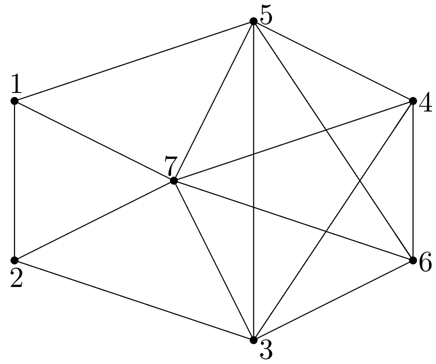


Figure 2: A perfect graph $G_2$ satisfying $N^2(G_2) \neq N_0^2(G_2)$

The same matrix in the proof of Claim 5 shows that $\bar{x} \in N_0^2(G_2)$. However, since $E(G_1) \subseteq E(G_2)$, $\text{FRAC}(G_2) \subseteq \text{FRAC}(G_1)$ and consequently $N^2(G_2) \subseteq N^2(G_1)$. Therefore, since $\bar{x} \notin N^2(G_1)$, $\bar{x} \notin N^2(G_2)$. The only facet of $N^2(G_2)$ that is not valid for $N_0^2(G_2)$ is $[3, 3, 1, 1, 1, 1, 4]x \leq 4$. Note that

$$N_0(G_2) = N(G_2) = \text{OC}(G_2) = \{x : x \text{ satisfies all triangle inequalities in } G_2\}.$$

Hence, $N$-$N_0$ Conjecture fails for perfect graphs as well. In the next section, we consider a weaker conjecture called the *Rank Conjecture*; as we remind the reader there, the Rank Conjecture holds for perfect graphs.

## 4 Polyhedral graph rank conjecture

Given $P \subseteq [0, 1]^n$, we define $r(P) := \min\left\{k \in \mathbb{Z}_+ : N^k(P) = P_I\right\}$ and call it the *$N$-rank of $P$*. We similarly define the *$N_0$-rank of $P$*, and denote it by $r_0(P)$. We start with an example in which $r(P) < r_0(P)$, where $P$ is lower-comprehensive.

**Example 7.** *Consider $P := \left\{x \in [0, 1]^3 : x_1 + x_2 \leq 1, 3x_1 + 4x_3 \leq 4, 4x_2 + 3x_3 \leq 4\right\}$. We define $U$:*

| $i$ | $j$ | $U_{ij}$ |
|---|---|---|
| $4x_2 + 3x_3 \leq 4$ | 2 | 1 |
| $3x_1 + 4x_3 \leq 4$ | 3 | 1 |
| $x_1 + x_2 \leq 1$ | 3 | $-3$ |

*Then*

$$U^T A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & -3 & 4 \end{pmatrix}$$

*and the inequality induced by $U$ is $x_1 + x_2 + x_3 \leq 1$, which implies that $N(P) = P_I$. However, the matrix*

$$\begin{pmatrix} 8 & 4 & 1 & 4 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 4 & 0 & 0 & 4 \end{pmatrix} \in M_0(P)$$

*shows that $\frac{1}{8}(4,1,4)^T \in N_0(P)$, and hence $N_0(P) \neq N(P)$. In this case (in contrast to Example 4), we have $r(P) = 1 < 2 = r_0(P)$.*

The $N_0$*-rank* (resp. $N$*-rank*) *of a graph* is the smallest $k$ such that $N_0^k(G) = \mathrm{STAB}(G)$ (resp. $N^k(G) = \mathrm{STAB}(G)$). Lipták and the second author [15] conjectured: $r_0(G) = r(G) \quad \forall$ graphs $G$. This *Rank Conjecture* is true for bipartite graphs, series-parallel graphs, perfect graphs and odd-star-subdivisions of graphs in $\mathcal{B}$ (defined in [15]—which contains cliques and wheels, among many other graphs). It is also true for antiholes and graphs that have $N_0$-rank $\leq 2$. Recently the first author proved that the Rank Conjecture holds for all 8-node graphs, and for 9-node graphs that contain a 7-hole or a 7-antihole as an induced subgraph [3]. However, the Rank Conjecture stays open. In particular, the question of "whether the *gaps* we see between $N_0^2(G)$ and $N^2(G)$ for the examples $G_1$, $G_2$ can be magnified for larger graphs to show that the Rank Conjecture also fails" remains open. Also, interesting in its own right is the further study of these gaps. For instance, upon defining the *gap* as

$$g_k(G) := \min\left\{\beta \in \mathbb{R} : \beta N^k(G) \supseteq N_0^k(G)\right\},$$

it is easy to verify for the examples of the previous section that $g_2(G_1) = g_2(G_2) = 1.05$. (Indeed, for every graph $G$, $g_0(G) = g_1(G) = g_{r_0(G)}(G) = 1$.) Characterizations of $g_k$ would be useful in other contexts (beyond the stable set polytope and including other lift-and-project operators) as well.

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