Complexity Analyses of Bienstock–Zuckerberg and Lasserre Relaxations on the Matching and Stable Set Polytopes*

Yu Hin Au and Levent Tunçel

Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1 Canada

Abstract. Many hierarchies of lift-and-project relaxations for 0,1 integer programs have been proposed, two of the most recent and strongest being those by Lasserre in 2001, and Bienstock and Zuckerberg in 2004. We prove that, on the LP relaxation of the matching polytope of the complete graph on $(2n+1)$ vertices defined by the nonnegativity and degree constraints, the Bienstock–Zuckerberg operator (even with positive semidefiniteness constraints) requires $\Theta(\sqrt{n})$ rounds to reach the integral polytope, while the Lasserre operator requires $\Theta(n)$ rounds. We also prove that Bienstock–Zuckerberg operator, without the positive semidefiniteness constraint requires approximately $n/2$ rounds to reach the stable set polytope of the n-clique, if we start with the fractional stable set polytope. As a by-product of our work, we consider a significantly strengthened version of Sherali–Adams operator and a strengthened version of Bienstock–Zuckerberg operator. Most of our results also apply to these stronger operators.

Keywords: matching polytope, lift-and-project methods, integer programming, semidefinite programming, convex relaxations

1 Introduction

Given a polytope $P \subseteq [0,1]^n$, we are interested in $P_I := \text{conv}\left\{P \cap \{0,1\}^n\right\}$, the *integer hull* of P. While it is impossible to efficiently find a description of P_I for a general P (unless $\mathcal{P} = \mathcal{NP}$), we may use properties that we know are satisfied by points in P_I to derive inequalities that are valid for P_I but not P.

Lift-and-Project methods provide a systematic way to generate a sequence of convex relaxations converging to the integer hull P_I. These methods have a special place in optimization as they lie at the intersection of combinatorial optimization and convex analysis (this goes back to work by Balas and others in the late 1960s and the early 1970s, see for instance Balas [Bal98] and the references therein). Some of the most attractive features of these methods are:

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- Convex relaxations of P_I obtained after $O(1)$ rounds of the procedure are *tractable* provided P is tractable (here tractable means that the underlying linear optimization problem is polynomial-time solvable).
- Many of these methods use *lifted* (higher dimensional) representations for the relaxations. Such representations sometimes allow compact (polynomial size in the input) convex representations of exponentially many facets.
- Most of these methods allow addition of positive semidefiniteness constraints in the lifted-space. This feature can make the relaxations much stronger in some cases, without sacrificing polynomial-time solvability. Moreover, these semidefiniteness constraints can represent an uncountable family of defining linear inequalities, such as those of the theta body of a graph.
- Systematic generation of tighter and tighter relaxations converging to P_I in at most n rounds makes the strongest of these methods good candidates for utilization in generating polynomial time approximation algorithms for hard problems, or for proving large integrality gaps (hence providing a negative result about approximability in the underlying hierarchy).

In the last two decades, many lift-and-project operators have been proposed (see [SA90], [LS91], [BCC93], [Las01] and [BZ04]), and have been applied to various discrete optimization problems. For instance, many families of facets of the stable set polytope of graphs are shown to be easily generated by these procedures [LS91] [LT03]. Also studied are their performances on set covering [BZ04], TSP relaxations [CD01] [Che05], max-cut [Lau02], and so on. For general properties of these operators and some comparisons among them, see [GT01], [HT08] and [Lau03].

In this paper, we focus on the strongest of the existing operators. Analyzing the behaviour of these strong operators on fundamental combinatorial optimization problems such as matching and the stable set problem, improves our understanding of these operators and their limitations. This in turn provides future research directions for further improvements of these hierarchies and related algorithms as well as the design and discovery of new ones.

Two of the strongest operators known to date are Las by Lasserre and BZ_+ by Bienstock and Zuckerberg. We are interested in these strongest operators because they provide the strongest tractable relaxations obtained this way. On the other hand, if we want to prove that some combinatorial optimization problem is difficult to attack by lift-and-project methods, then we would hope to establish them on the strongest existing hierarchy for the strongest negative results. For example, some of the known non-approximability results on vertex cover are based on Lovász and Schrijver's LS_+ operator ([GMPT06], [STT06]), which is known not to be the strongest. By understanding the more powerful operators (or better yet, inventing new ones), we could either obtain better approximations for vertex cover (and other hard problems), or lay the groundwork for yet stronger non-approximability results.

Our most striking findings are on the matching and stable set polytopes. Stephen and the second author [ST99] proved that LS_+ requires n rounds on the matching polytope of the $(2n+1)$-clique, establishing the first bad instance for LS_+ since it was proposed in 1991. Subsequently, some other lift-and-project operators have been shown to also perform poorly in this instance. For the Balas–Ceria–Cornuéjols operator [BCC93], Aguilera, Bianchi and Nasini [ABN04] showed that n^2 rounds are needed. More recently, Mathieu and Sinclair [MS09] proved that Sherali–Adams operator requires $(2n-1)$ rounds. The related question for the Lasserre operator has been open since 2001, and for the BZ_+ operator since 2003. We answer these questions as $\Theta(n)$ rounds and $\Theta(\sqrt{n})$ rounds respectively. This establishes the first example on which BZ_+ requires more than $O(1)$ rounds to reach the integer hull. For some bad instances for Lasserre's operator, see [Lau02] and [Che07]. An implication of our results is that all of these procedures become exponential time algorithms on the matching problem (assuming that they are implemented as stated). As a by-product of our analysis, we develop some new tools and modify some existing ones in the area. We also construct a very strong version of Sherali–Adams operator that we call SA_+ (there are other weaker versions of SA_+ in the recent literature called *Sherali–Adams SDP*, see [BGM10] and the references therein) and relate it to the BZ_+ operator. Also, as another by-product of our approach we strengthen the BZ, BZ_+ operators (our analyses also applies to these stronger versions). We conclude the paper by proving that the BZ operator requires approximately $n/2$ rounds to compute the stable set polytope of the n-clique.

2 Preliminaries

For convenience, we denote $\{0,1\}^n$ by $\mathcal{F}$ and the set $\{1,2,\ldots,n\}$ by $[n]$ herein. Given $x \in [0,1]^n$, let $\hat{x}$ denote the vector $\begin{pmatrix}1\\x\end{pmatrix}$ in $\mathbb{R}^{n+1}$, where the new coordinate is indexed by 0. Let e_i denote the i^{th} unit vector, and for any square matrix M let $\mathrm{diag}(M)$ denote the vector formed by the diagonal entries of M. Next, given $P \subseteq [0,1]^n$, define the cone

$$K(P) := \left\{ \begin{pmatrix}\lambda\\ \lambda x\end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^n : \lambda \geq 0, x \in P \right\}.$$

Define $\mathcal{A} := 2^{\mathcal{F}}$. For each $x \in \mathcal{F}$, we define the vector $x^{\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$ such that

$$x^{\mathcal{A}}_\alpha = \begin{cases} 1 \text{ if } x \in \alpha \\ 0 \text{ otherwise.} \end{cases}$$

I.e., each coordinate of $\mathcal{A}$ can be interpreted as a subset of the vertices of the n-dimensional hypercube, and $x^{\mathcal{A}}_\alpha = 1$ if and only if the point x is contained in the set α. It is not hard to see that for all $x \in \mathcal{F}$ and $i \in [n]$, we have $x^{\mathcal{A}}_{\mathcal{F}} = 1$, and $x^{\mathcal{A}}_{\{y \in \mathcal{F} : y_i = 1\}} = x_i$. Another important property of $x^{\mathcal{A}}$ is that, given disjoint subsets $\alpha_1, \alpha_2, \ldots, \alpha_k \subseteq \mathcal{F}$, we know that

$$x^{\mathcal{A}}_{\alpha_1} + x^{\mathcal{A}}_{\alpha_2} + \cdots + x^{\mathcal{A}}_{\alpha_k} \leq 1, \tag{1}$$

and equality holds if $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ partitions $\mathcal{F}$. Let $\mathbb{S}^n$ denote the set of n-by-n real, symmetric matrices and $\mathbb{S}^n_+ \subset \mathbb{S}^n$ denote the set of symmetric, positive semidefinite n-by-n matrices. For any given $x \in \mathcal{F}$, if we define $Y^x_{\mathcal{A}} := x^{\mathcal{A}}(x^{\mathcal{A}})^T$, then we know that the entries of $Y^x_{\mathcal{A}}$ have considerable structure. Most notably, the following must hold:

- $Y^x_{\mathcal{A}} \in \mathbb{S}^{\mathcal{A}}_+$; $Y^x_{\mathcal{A}} e_{\mathcal{F}} = (Y^x_{\mathcal{A}})^T e_{\mathcal{F}} = \operatorname{diag}(Y^x_{\mathcal{A}}) = x^{\mathcal{A}}$; $Y^x_{\mathcal{A}} e_\alpha \in \left\{0, x^{\mathcal{A}}\right\}, \ \forall \alpha \in \mathcal{A}$;
- $Y^x_{\mathcal{A}}[\alpha, \beta] = 1 \iff x \in \alpha \cap \beta$
- If $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, then $Y^x_{\mathcal{A}}[\alpha_1, \beta_1] = Y^x_{\mathcal{A}}[\alpha_2, \beta_2]$.

Zuckerberg [Zuc03] showed that most of the existing lift-and-project operators can be interpreted under the common theme of placing constraints that are relaxations of the above conditions on submatrices of $Y^x_{\mathcal{A}}$. In the remainder of this section, we define the operators proposed by Lasserre [Las01] and Bienstock–Zuckerberg [BZ04]. However, it is helpful to look at a strengthened version of the Sherali–Adams' operator [SA90] first, which has an additional positive semidefiniteness constraint. (We denote the new operator by SA_+.) Our SA_+ is similar to an operator studied by Benabbas, Georgiou and Magen [BGM10] and others, even though our version is stronger. The SA_+ operator will be useful in simplifying our analysis and improving our understanding of the Bienstock–Zuckerberg operator. For ease of reference, Figure 1 provides a glimpse of the relative strengths of various known lift-and-project operators. Each arrow in the chart denotes "is refined by" (i.e. the operator that is at the head of an arrow is stronger than that at the tail). Also, operators whose labels are framed in a box are new (the operators BZ' and BZ'_+ will be defined in Section 2.3).

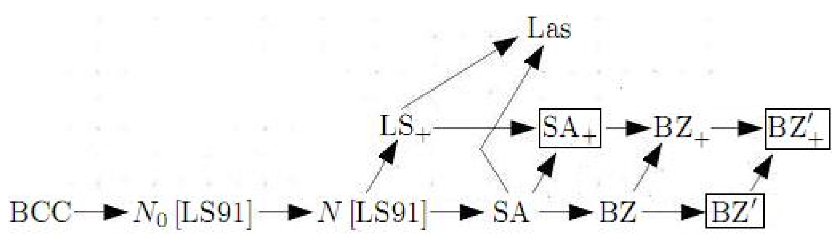

Fig. 1. A strength chart of lift-and-project operators

While all of these operators can be applied to any polytope contained in the unit hypercube (and in the Lasserre operator's case, sets defined by polynomial inequalities), we will focus our discussion on their application to lower-comprehensive polytopes (polytopes P such that $u \in P$, $0 \leq v \leq u$ implies $v \in P$), since our main objects of interest are the matching and stable set polytopes.

2.1 The SA and SA_+ operator

Given a set of indices $S \subseteq [n]$ and $t \in \{0, 1\}$, we define

$$S|_t := \{x \in \mathcal{F} : x_i = t, \ \forall i \in S\}. \tag{2}$$

To reduce cluttering, we write $i|_t$ instead of $\{i\}|_t$. Also, for ease of reference, given any $\alpha \in \mathcal{A}$ in the form of $S|_1 \cap T|_0$ where $S, T \subseteq [n]$ are disjoint, we call S the set of *positive indices* in α, T the set of *negative indices* in α, and $|S| + |T|$ the *order* of α. Finally, for any integer $i \in [0, n]$, define $\mathcal{A}_i := \{S|_1 \cap T|_0 : S, T \subseteq [n], S \cap T = \emptyset, |S| + |T| \leq i\}$ and $\mathcal{A}_i^+ := \{S|_1 : S \subseteq [n], |S| \leq i\}$. Given a fixed integer $k \in [1, n]$, the SA^k and SA_+^k operators can be defined as follows:

1. Let $\tilde{\mathrm{SA}}^k(P)$ denote the set of matrices $Y \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_k}$ that satisfy all of the following conditions:

(SA 1) $Y[\mathcal{F}, \mathcal{F}] = 1$;

(SA 2) $\hat{x}(Ye_\alpha) \in K(P)$ for every $\alpha \in \mathcal{A}_k$;

(SA 3) For each $S|_1 \cap T|_0 \in \mathcal{A}_{k-1}$, impose

$$Ye_{S|_1 \cap T|_0} = Ye_{S|_1 \cap T|_0 \cap j|_1} + Ye_{S|_1 \cap T|_0 \cap j|_0}, \ \forall j \in [n] \setminus (S \cup T).$$

(SA 4) For each $\alpha \in \mathcal{A}_1^+, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta = \emptyset$, impose $Y[\alpha, \beta] = 0$;

(SA 5) For every $\alpha_1, \alpha_2 \in \mathcal{A}_1^+, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, impose $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

2. Let $\tilde{\mathrm{SA}}_+^k(P)$ denote the set of matrices $Y \in \mathbb{S}_+^{\mathcal{A}_k}$ that satisfies all of the following conditions:

(SA_+ 1) (SA 1), (SA 2) and (SA 3);

(SA_+ 2) For each $\alpha, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta \cap P = \emptyset$, impose $Y[\alpha, \beta] = 0$;

(SA_+ 3) For any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, impose $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

3. Define
$$\mathrm{SA}^k(P) := \left\{x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{SA}}^k(P) : Ye_{\mathcal{F}} = \hat{x}\right\}$$
and
$$\mathrm{SA}_+^k(P) := \left\{x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{SA}}_+^k(P) : \hat{x}(Ye_{\mathcal{F}}) = \hat{x}\right\}.$$

The SA_+^k operator extends the lifted space of the SA^k operator to a set of square matrices, and imposes an additional positive semidefiniteness constraint. Moreover, SA_+^k refines the LS_+^k operator devised by Lovász and Schrijver in [LS91], which we define below. Given $P \subseteq [0, 1]^n$,

$$\begin{aligned} \mathrm{LS}_+(P) := \{x \in \mathbb{R}^n : \exists Y \in \mathbb{S}_+^{n+1}; Ye_0 = \mathrm{diag}(Y) = \hat{x}; \\ Ye_i, Y(e_0 - e_i) \in K(P), \ \forall i \in [n]\}. \end{aligned}$$

For any integer $k \geq 1$, define $\mathrm{LS}_+^k(P) := \mathrm{LS}_+(\mathrm{LS}_+^{k-1}(P))$, where $\mathrm{LS}_+^0(P) := P$. Then we have the following:

Proposition 1 *For every polytope $P \subseteq [0, 1]^n$ and every integer $k \geq 1$, $\mathrm{SA}_+^k(P) \subseteq \mathrm{LS}_+(\mathrm{SA}_+^{k-1}(P))$.*

It follows immediately from Proposition 1 that $\mathrm{SA}_+^k(P) \subseteq \mathrm{LS}_+^k(P)$. We also remark that the condition (SA_+ 2) can be efficiently checked. For $\alpha, \beta \in \mathcal{A}_k, \alpha = S|_1 \cap T|_0$ and $\beta = S'|_1 \cap T'|_0$, $\alpha \cap \beta \cap P = \emptyset$ if and only if $\chi^{S \cup S'} \notin P$, since P is lower-comprehensive.

2.2 The Lasserre operator

We next turn our attention to Lasserre's operator defined in [Las01], denoted Las^k herein. Our presentation of the operator is closer to that in [Lau03]. Given $P := \{x \in [0,1]^n : Ax \leq b\}$, and an integer $k \in [n]$,

1. Let $\tilde{\mathrm{Las}}^k(P)$ denote the set of matrices $Y \in \mathbb{S}_+^{\mathcal{A}_{k+1}^+}$ that satisfy all of the following conditions:

(Las 1) $Y[\mathcal{F}, \mathcal{F}] = 1$;

(Las 2) For each $j \in [m]$, let A^j be the j^{th} row of A. Define the matrix $Y^j \in \mathbb{S}^{\mathcal{A}_k^+}$ such that

$$Y^j[S|_1, S'|_1] := b_j Y[S|_1, S'|_1] - \sum_{i=1}^{n} A_i^j Y[(S \cup \{i\})|_1, (S' \cup \{i\})|_1]$$

and impose $Y^j \succeq 0$.

(Las 3) For every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k^+$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, impose $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

2. Define

$$\mathrm{Las}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{Las}}^k(P) : \hat{x}(Ye_{\mathcal{F}}) = \hat{x} \right\}.$$

In our setting, the Las-rank of a polytope P (the smallest k such that $\mathrm{Las}^k(P) = P_I$) is equal to the Theta-rank, defined in [GPT10].

2.3 The Bienstock-Zuckerberg operators

Finally, we look into the operators devised by Bienstock and Zuckerberg [BZ04]. One of the features of their algorithms is that they use variables in $\mathcal{A}$ that were not exploited by the operators proposed earlier, in conjunction with some new constraints. We denote their algorithms by BZ and BZ_+, but we also present modified variants called BZ' and BZ'_+. These modified algorithms have the advantage of being stronger; moreover, they are simpler to present. We discuss this in more detail after stating the algorithm.

Suppose we are given $P := \{x \in [0,1]^n : Ax \leq b\}$. The BZ' algorithm can be viewed as a two-step process. The first step is *refinement*. Recall that A^i is the i-th row of A. If $O \subseteq [n]$ satisfies

$$O \subseteq \mathrm{supp}(A^i); \sum_{j \in O} A_j^i > b_i; |O| \leq k \text{ or } |O| \geq |\mathrm{supp}(A^i)| - k,$$

for some $i \in [m]$, then we call O a *k-small obstruction*. (Here, $\mathrm{supp}(a)$ denotes the support of a, i.e., the index set of nonzero components of a.) Let $\mathcal{O}$ denote the set of all k-small obstructions of P (or more precisely, of the system $Ax \leq b$). Notice that, for any obstruction $O \in \mathcal{O}$, and for every integral vector $x \in P$, the inequality $\sum_{i \in O} x_i \leq |O| - 1$ holds. Thus,

$$P_{\mathcal{O}} := \left\{ x \in P : \sum_{i \in O} x_i \leq |O| - 1, \ \forall O \in \mathcal{O} \right\}$$

is a relaxation of the integer hull of P that is potentially tighter than P. We call $P_{\mathcal{O}}$ the *k-refinement* of P.

The second step of the algorithm is *lifting*. Before we give the details of this step, we need another intermediate collection of sets of indices, called *walls*. We call $W \subseteq [n]$ a wall if either $W = \{i\}$ for some $i \in [n]$, or if there exist $\ell \leq k$ distinct obstructions $O_1, \dots, O_\ell \in \mathcal{O}$ such that $W = \bigcup_{i,j \in [\ell], i \neq j} (O_i \cap O_j)$. That is, each subset of up to k obstructions generate a wall, which is the set of elements that appear in at least two of the given obstructions.

Next, we define the collection of *tiers*

$$\mathcal{S} := \left\{ S \subseteq [n] : \exists W_{i_1}, \dots, W_{i_{k-1}} \in \mathcal{W}, S \subseteq \bigcup_{j=1}^{k-1} W_{i_j} \right\}.$$

I.e., we include a set of indices S as a tier if there exist $k-1$ walls whose union contains S. Note that the empty set and the singleton-sets are always tiers.

Finally, for any set $U \subseteq [n]$ and a nonnegative integer r, we define

$$U|_{<r} := \left\{ x \in \{0,1\}^n : \sum_{i \in U} x_i \leq r - 1 \right\}. \tag{3}$$

We will see that the elements in $\mathcal{A}$ that are being generated by BZ$'$ all take the form $S|_1 \cap T|_0 \cap U|_{<r}$, where S, T, U are disjoint sets of indices. For a set α in this form, we let S (resp. T) denote the set of positive (resp. negative) indices of α, and define the order of α to be $|S| + |T| + |U|$. We are now ready to describe the lifting step:

1. Define $\mathcal{A}'$ to be the set consisting of the following: For each tier $S \in \mathcal{S}$ and each $T \subseteq S$ such that $|T| \leq k-1$, include the sets
$$(S \setminus T)|_1 \cap T|_0; \tag{4}$$
if $|T| < k-1$, and $U \subseteq S \setminus T$ such that $|U| + |T| > k-1$, then also include
$$(S \setminus (T \cup U))|_1 \cap T|_0 \cap U|_{<|U|-(k-1-|T|)}, \tag{5}$$

2. Let $\tilde{\mathrm{BZ}}'^k(P)$ denote the set of matrices $Y \in \mathbb{S}^{\mathcal{A}'}$ that satisfy all of the following conditions:

(BZ$'$ 1) $Y[\mathcal{F}, \mathcal{F}] = 1$;

(BZ$'$ 2) For any column x of the matrix Y,
 (i) $0 \leq x_\alpha \leq x_{\mathcal{F}}$, for all $\alpha \in \mathcal{A}'$;
 (ii) $\hat{x}(x) \in K(P_{\mathcal{O}})$;
 (iii) $x_{i|_1} + x_{i|_0} = x_{\mathcal{F}}$ for every $i \in [n]$;
 (iv) For each $\alpha \in \mathcal{A}'$ in the form of $S|_1 \cap T|_0$ impose the inequalities

$$x_{i|_1} \geq x_\alpha, \quad \forall i \in S, \tag{6}$$
$$x_{i|_0} \geq x_\alpha, \quad \forall i \in T, \tag{7}$$
$$x_\alpha + x_{(S \cup \{i\})|_1 \cap (T \setminus \{i\})|_0} = x_{S|_1 \cap (T \setminus \{i\})|_0}, \quad \forall i \in T, \tag{8}$$
$$\sum_{i \in S} x_{i|_1} + \sum_{i \in T} x_{i|_0} - x_\alpha \leq (|S| + |T| - 1) x_{\mathcal{F}}. \tag{9}$$

For each $\alpha \in \mathcal{A}'$ in the form $S|_1 \cap T|_0 \cap U|_{<r}$, impose the inequalities

$$x_{i|_1} \geq x_\alpha, \quad \forall i \in S, \tag{10}$$

$$x_{i|_0} \geq x_\alpha, \quad \forall i \in T, \tag{11}$$

$$\sum_{i \in U} x_{i|_0} \geq (|U| - (r-1)) x_\alpha, \tag{12}$$

$$x_\alpha = x_{S|_1 \cap T|_0} - \sum_{U' \subseteq U, |U'| \geq r} x_{(S \cup U')|_1 \cap (T \cup (U \setminus U'))|_0}. \tag{13}$$

(BZ′ 3) For each pair $\alpha, \beta \in \mathcal{A}'$, if $\alpha \cap \beta \cap P = \emptyset$, then impose $Y[\alpha, \beta] = 0$;
(BZ′ 4) For variables $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathcal{A}'$, if $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, then impose $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

– Define

$$\mathrm{BZ}'^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{BZ}}'^k(P) : \hat{x}(Ye_0) = \hat{x} \right\},$$

and

$$\mathrm{BZ}'^k_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{BZ}}'^k(P) \cap \mathbb{S}_+^{\mathcal{A}'} : \hat{x}(Ye_0) = \hat{x} \right\}.$$

Similar to the case of SA^k, BZ'^k can be seen as creating columns that correspond to sets that partition $\mathcal{F}$. While SA^k only generates a partition for each subset of up to k indices, BZ'^k does so for every tier, which is a much broader collection of indices. For a tier S up to size k, it does the same as SA^k and generates $2^{|S|}$ columns corresponding to all possible negations of indices in S. However, for S of size greater than k, it generates a "k-deep" partition of S: a column for $(S \setminus T)|_1 \cap T|_0$ for each $T \subseteq S$ of size up to $k-1$, and the column $S|_{<|S|-k+1}$. Moreover, it also generates columns that partition $(S \setminus T)|_1 \cap T|_0$ for every tier S and every $T \subseteq S$ such that $|T| < k-1$: For each $U \subseteq S$ that is disjoint from T such that $|T| + |U| > k-1$, the algorithm introduces the columns

$$(S \setminus T)|_1 \cap T|_0 \cap (U \setminus U')|_1 \cap U'|_0$$

for all U' of size $\leq (k-1) - |T|$ (so the total number of the negative indices does not exceed $k-1$). It also generates a column for

$$(S \setminus T)|_1 \cap T|_0 \cap U|_{<|U|-(k-1-|T|)}$$

to capture the remainder of the partition.

Notice that in BZ′, we have generated exponentially many variables, whereas in the original BZ only polynomially many are selected. The role of walls are also much more important in selecting the variables in BZ, which we have intentionally suppressed in BZ′ to make our presentation and analysis easier. We will only use these modified operators to establish negative results, so that the same bounds apply to the original Bienstock–Zuckerberg operators, details of which will be in the full version of this extended abstract.

The main result Bienstock and Zuckerberg achieved with the BZ^k algorithm is when it is applied to set covering problems. Given an inequality $a^T x \geq a_0$ such that $a \geq 0$ and $a_0 > 0$, its *pitch* is defined to be the smallest $j \in \mathbb{N}$ such that

$$S \subseteq \mathrm{supp}(a), |S| \geq j \Rightarrow a^T \chi^S \geq a_0.$$

Also, let $\bar{e}$ denote the all-ones vector of suitable size. Then they showed the following powerful result:

Theorem 2 *Suppose $P := \{x \in [0,1]^n : Ax \geq \bar{e}\}$ where A is a $0,1$ matrix. Then for every $k \geq 2$, every valid inequality of P_I that has pitch at most k is valid for $\mathrm{BZ}^k(P)$.*

Note that all inequalities whose coefficients are integral and at most k have pitch no more than k.

2.4 Matching Polytope and the notion of rank

We next define the matching polytope of graphs. Given a simple, undirected graph $G = (V, E)$, we define

$$MT(G) := \left\{ x \in [0,1]^E : \sum_{j:\{i,j\} \in E} x_{ij} \leq 1, \ \forall i \in V \right\}.$$

Then the integral points in $MT(G)$ are exactly the incidence vectors of the matchings of G. For any lift-and-project operator Γ, we abbreviate $\Gamma(MT(G))$ as $\Gamma(G)$. Also, for any polytope P, we define the *Γ-rank of P* to be the smallest integer k such that $\Gamma^k(P) = P_I$. The notion of rank gives us a measure of how close P is to P_I with respect to Γ. Moreover, it is useful when comparing the performance of different operators.

3 Some tools for upper bound analyses

In this section, we present some intermediate results that will help us establish our main results. These tools could be useful in analyzing the lift-and-project ranks of other polytopes as well. Given $j \in \mathbb{Z}_+$, let $[n]_j$ denote all subsets of $[n]$ of size j. Suppose $Y \in \mathbb{S}^{\mathcal{A}'}$ for some $\mathcal{A}' \subseteq \mathcal{A}$. We say that Y is *ℓ-established* if all of the following conditions hold:

(ℓ1) $Y[\mathcal{F}, \mathcal{F}] = 1$;
(ℓ2) $Y \succeq 0$;
(ℓ3) $\mathcal{A}_\ell^+ \subseteq \mathcal{A}'$;
(ℓ4) For any $\alpha, \beta, \alpha', \beta' \in \mathcal{A}_\ell^+$ such that $\alpha \cap \beta = \alpha' \cap \beta', Y[\alpha, \beta] = Y[\alpha', \beta']$.
(ℓ5) For any $\alpha, \beta \in \mathcal{A}_\ell^+, Y[\mathcal{F}, \beta] \geq Y[\alpha, \beta]$.

Notice that any matrix $Y \in \tilde{\mathrm{SA}}_+^\ell(P)$ is ℓ-established. A matrix in the lifted space of BZ'_+ is also ℓ-established if all subsets of size up to ℓ are generated as tiers. Given such a matrix, we may define $\mathcal{M} := \bigcup_{i=0}^{2\ell} [n]_i$ and $y \in \mathbb{R}^{\mathcal{M}}$ such that $y_S = Y[S'|_1, S''|_1]$, where S', S'' are subsets of $[n]$ of size at most ℓ such that $S' \cup S'' = S$. Notice that by (ℓ4), the value of y_S does not depend on the choice of S', S''. Finally, we define $Z \in \mathbb{R}^{\{0\} \cup [2\ell]}$ such that

$$Z_i := \sum_{S \subseteq [n]_i} y_S, \ \forall i \geq 0.$$

By (ℓ1), Z_0 is always equal to 1. Also note that, $Z_1 = \sum_{i=1}^n Y[i|_1, \mathcal{F}]$. We see that the entries of Z are related to each other. For example, if $\hat{x}(Ye_{\mathcal{F}})$ is an integral $0, 1$ vector, then by (ℓ5) we know that $y_S \leq 1$ for all S, and $y_S > 0$ only if $y_{\{i\}} = 1, \ \forall i \in S$. Thus, we can infer that

$$Z_j = \sum_{S \in [n]_j} y_S \leq \binom{Z_1}{j}, \ \forall j \in [2\ell].$$

Next, we show that the positive semidefiniteness of Y also forces the Z_i's to relate to each other, somewhat similarly to the above. The following result would be more intuitive by noting that $\binom{p}{i+1} / \binom{p}{i} = \frac{p-i}{i+1}$.

Proposition 3 *Suppose $Y \in \mathbb{S}^{\mathcal{A}'}$ is ℓ-established, and y, Z are defined as above. If there exists $p \in \mathbb{R}_+$ such that*

$$Z_{i+1} \leq \left(\frac{p-i}{i+1} \right) Z_i, \ \forall i \in [\ell, 2\ell - 1],$$

then $Z_i \leq \binom{p}{i}, \ \forall i \leq 2\ell$. In particular, $Z_1 \leq p$.

An immediate but noteworthy implication of Proposition 3 is the following:

Corollary 4 *Suppose $Y \in \mathbb{S}^{\mathcal{A}'}$ is ℓ-established, and y, Z are defined as above. If there exists $p \in [0, \ell]$ such that $Z_i = 0, \ \forall i > p$, then $Z_1 \leq p$.*

4 The SA_+-rank, the Las-rank, and the Theta-rank of the Matching Polytope

We now turn to our main results and determine the lift-and-project ranks of $MT(K_{2n+1})$ for various operators. First, we study the SA_+-rank.

Theorem 5 *The SA_+-rank of $MT(K_{2n+1})$ is at least $\left\lfloor \frac{n}{2} \right\rfloor + 1$.*

Proof (sketch). We prove our claim by showing that $\frac{1}{4n}\bar{e} \in \mathrm{SA}_+^n(K_{4n+1})$, implying that $MT(K_{4n+1})$ has SA_+-rank at least $n+1$, from which our assertion follows. Define $Y \in \mathbb{S}^{\mathcal{A}_n}$ such that $Y[\emptyset, \emptyset] := 1$, and $Y[S_1|_1, S_2|_1] :=$

$\prod_{i=1}^{|S_1 \cup S_2|} \frac{1}{4n+2-2i}$ if $S_1 \cup S_2$ is a matching and 0 otherwise. Also, set $Y[S_1|_1 \cap T_1|_0, S_2|_1 \cap T_2|_0] := \sum_{U \subseteq T_1 \cup T_2} (-1)^{|U|} Y[S_1 \cup (U \cap T_1)|_1, S_2 \cup (U \cap T_2)|_1]$.

Notice that $\bar{x}(Ye_{\mathcal{F}}) = \frac{1}{4n}\bar{e}$, and (SA 1), (SA 3), (SA$_+$ 2) and (SA$_+$ 3) all hold by the construction of Y. Also, it was shown in [MS09] that (SA 2) holds. Thus, it only remains to verify that $Y \succeq 0$. By exploiting the linear dependencies of the columns of Y and the symmetries of the complete graph, the task of showing $Y \succeq 0$ can be reduced to showing $Y' \succeq 0$, where

$$Y'[i,j] := \sum_{\substack{S_1, S_2 \subseteq E, \\ |S_1|=i, |S_2|=j}} Y[S_1|_1, S_2|_1], \ \forall i,j \in \{0,1,\ldots,n\}\,.$$

It can be checked that $Y'[i,j] = \binom{\frac{4n+1}{2}}{i}\binom{\frac{4n+1}{2}}{j}$ for all integers $i,j \in [0,n]$. Hence $Y' = (Y'e_0)(Y'e_0)^T$ and our claim follows. $\square$

Next, we employ the upper bound proving techniques from Section 3 and the notion of ℓ-established to prove the next result.

Proposition 6 *The* SA$_+$*-rank of $MT(K_{2n+1})$ is at most $n - \left\lfloor \frac{\sqrt{2n+1}-1}{2} \right\rfloor$.*

Somewhat surprisingly, a lower bound of the Las-rank of the matching polytope follows almost immediately from the proof of Theorem 5.

Theorem 7 *The* Las*-rank and Theta-rank of $MT(K_{2n+1})$ is at least $\left\lfloor \frac{n}{2} \right\rfloor$ and at most n.*

5 The BZ$_+$-rank of the Matching Polytope

Next, we turn to the BZ$_+$-rank of the matching polytope. Before we do that, it is beneficial to characterize some of the variables generated by the stronger BZ$'^k_+$ that obviously do not help generate any cuts. We say that a tier S generated by BZ$'^k$ is *ℓ-useless* if

1. For all $T \subseteq S$ such that $|T| \leq k-1$, $(S \setminus T)|_1 \cap P = \emptyset$;
2. $\sum_{i \in S} x_i \leq |S| - k$ is valid for SA$^\ell(P_{\mathcal{O}})$.

Then, we have the following:

Lemma 8 *Suppose there exists $\ell \in \mathbb{Z}_+$ such that all tiers S generated by* BZ$'^k$ *of size greater than ℓ are ℓ-useless. Then*

$$\mathrm{BZ}'^k(P) \supseteq \mathrm{SA}^{2\ell}(P_{\mathcal{O}}) \ \text{and} \ \mathrm{BZ}'^k_+(P) \supseteq \mathrm{SA}^\ell_+(P_{\mathcal{O}}).$$

We are now ready to approximate the BZ$_+$-rank of $MT(K_{2n+1})$, to within a constant factor.

Theorem 9 *Suppose $G = K_{2n+1}$. Then the* BZ$_+$*-rank of $MT(G)$ is between $\sqrt{n}$ and $\sqrt{2n}+1$.*

In fact, we prove that the above lower bound applies to the stronger BZ′. We also remark that, in general, adding redundant inequalities to the system $Ax \leq b$ would generate more obstructions and walls, and thus could improve the performance of BZ_+. In fact, if we let $G := K_{2n+1}$ and include every valid inequality of $MT(G)$ in the initial description of the polytope $MT(G)$, then any matrix $Y \in \tilde{\mathrm{BZ}}_+^2(G)$ is actually n-established, which implies that $\hat{x}(Ye_{\mathcal{F}}) \in K(MT(G)_I)$.

6 The BZ-rank of the Stable Set Polytope

Another family of polytopes related to graphs that has been studied extensively is the family of stable set polytopes. Given a graph $G = (V, E)$, its *fractional stable set polytope* is defined to be

$$FRAC(G) := \left\{ x \in [0,1]^V \ : \ x_i + x_j \leq 1, \ \forall \{i,j\} \in E \right\}.$$

Then the *stable set polytope* $STAB(G) := FRAC(G)_I$ is precisely the convex hull of incidence vectors of stable sets of G. For the complete graph $G := K_n$, $FRAC(G)$ is known to have rank 1 with respect to the LS_+ and Las operators. Proposition 1 implies that it also has SA_+-rank 1. Its BZ_+-rank is 1 as well, as it is not hard to see that SA_+^1 is refined by BZ_+^1. However, the rank is known to be $\Theta(n)$ for all other operators that yield only polyhedral relaxations, such as SA and Lovász–Schrijver's N operator [LS91]. We show that BZ operator also has the same property.

Theorem 10 *Suppose G is the complete graph on $n \geq 5$ vertices. Then the* BZ*-rank of $FRAC(G)$ is either $\left\lceil \frac{n}{2} - 1 \right\rceil$ or $\left\lceil \frac{n}{2} \right\rceil$.*

Again, the lower bound of the above also applies to BZ′.

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