ELEMENTARY POLYTOPES WITH HIGH LIFT-AND-PROJECT RANKS FOR STRONG POSITIVE SEMIDEFINITE OPERATORS

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Abstract. We consider operators acting on convex subsets of the unit hypercube. These operators are used in constructing convex relaxations of combinatorial optimization problems presented as a 0,1 integer programming problem or a 0,1 polynomial optimization problem. Our focus is mostly on operators that, when expressed as a lift-and-project operator, involve the use of semidefiniteness constraints in the lifted space, including operators due to Lasserre and variants of the Sherali–Adams and Bienstock–Zuckerberg operators. We study the performance of these semidefinite-optimization-based lift-and-project operators on some elementary polytopes — hypercubes that are chipped (at least one vertex of the hypercube removed by intersection with a closed halfspace) or cropped (all 2^n vertices of the hypercube removed by intersection with 2^n closed halfspaces) to varying degrees of severity ρ. We prove bounds on ρ where the Sherali–Adams operator (strengthened by positive semidefiniteness) and the Lasserre operator require n iterations to compute the integer hull of the aforementioned examples, as well as instances where the Bienstock–Zuckerberg operators require $\Omega(\sqrt{n})$ iterations to return the integer hull of the chipped hypercube. We also show that the integrality gap of the chipped hypercube is invariant under the application of several lift-and-project operators of varying strengths.

1. Introduction

A foundational tool in tackling many combinatorial optimization problems is the construction of convex relaxations. Starting with a 0,1 integer programming formulation of the given problem, the goal is to find a tractable (whether in practice or theory, hopefully in both) optimization problem with essentially the same linear objective function, but a convex feasible region. Let $P \subseteq [0,1]^n$ denote the feasible region of the linear programming relaxation of an initial 0,1 integer programming problem. In our convex relaxation approach, we are hoping to construct a tractable representation of the convex hull of integer points in P, i.e., the *integer hull* of P

$$P_I := \operatorname{conv}\left(P \cap \{0,1\}^n\right).$$

However, it is impossible to efficiently find a tractable description of P_I for a general P (unless $\mathcal{P} = \mathcal{NP}$). So, in many cases we may have to be content with tractable convex relaxations that are not exact (i.e. strict supersets of the integer hull of P).

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Lift-and-project methods provide an organized way of generating a sequence of convex relaxations of P which converge to the integer hull P_I of P in at most n rounds. Minimum number of rounds required to obtain the integer hull by a lift-and-project operator Γ is called the Γ*-rank* of P. The computational success of lift-and-project methods on some combinatorial optimization problems and various applications is relatively well documented (starting with the theoretical foundations in Balas' work in the 1970s [Bal74]; this appeared as [Bal98]), and the majority of these computational successes come from lift-and-project methods which generate polyhedral relaxations. While many lift-and-project methods utilize in addition positive semidefiniteness constraints which in theory help generate tighter relaxations of P_I, the underlying convex optimization problems require significantly more computational resources to solve, and are prone to run into more serious numerical stability issues. Therefore, before committing to the usage of a certain lift-and-project method, it would be wise to understand the conditions under which the usage of additional computational resources would be well justified. Indeed, this argument applies to any collection of lift-and-project operators that trade off quality of approximation with computational resources (time, memory, etc.) required. That is, to utilize the strongest operators, one needs a better understanding of the class of problems on which these strongest operators' computational demands will be worthwhile in the returns they provide.

In the next section, we introduce a number of known lift-and-project operators and some of their basic properties, with the focus being on the following operators (all of which utilize positive semidefiniteness constraints):

- SA_+ (see [Au14, AT16]), a positive semidefinite variant of the Sherali–Adams operator SA defined in [SA90];
- Las, due to Lasserre [Las01];
- BZ'_+ (see [Au14, AT16]), a strengthened version of the Bienstock–Zuckerberg operator BZ_+ [BZ04].

Then, in Section 3, we look into some elementary polytopes which represent some basic situations in 0,1 integer programs. We consider two families of polytopes: unit hypercubes that are *chipped* or *cropped* to various degrees of severity. First, given an integer $n \geq 1$ and a real number ρ where $0 \leq \rho \leq n$, the *chipped hypercube* is defined to be

$$P_{n,\rho} := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq n - \rho \right\}.$$

Similarly, we define the *cropped hypercube*

$$Q_{n,\rho} := \left\{ x \in [0,1]^n : \sum_{i \in S} (1 - x_i) + \sum_{i \notin S} x_i \geq \rho, \ \forall S \subseteq [n] \right\},$$

where $[n]$ denotes the set $\{1, \ldots, n\}$. These two families of polytopes have been shown to be bad instances for many lift-and-project methods and cutting-plane procedures (see, among others, [CCH89, CL01, CD01, GT01, Lau03, Che07, PS10, DP14, KLM15, BDG17]). Moreover, these elementary sets are interesting in many other contexts as well. For instance, note that each constraint defining $Q_{n,\rho}$ removes a specific extreme point of the unit hypercube from the feasible region. In many 0,1 integer programming problems and in 0,1 mixed integer programming problems, such *exclusion* constraints are relatively commonly used.

Herein, we show that these sets are also bad instances for the strongest known operators, extending the previously known results in this vein. In particular, we show the following:

- The SA_+-rank of $P_{n,\rho}$ is n for all $\rho \in (0,1)$, and is at most $n - \lceil \rho \rceil + 1$ for all $\rho \in (0,n)$. In contrast, we show that $\tilde{\mathrm{LS}}$ (a simple polyhedral operator defined in [GT01] that is similar to the LS_0 operator due to Lovász and Schrijver [LS91]) requires n iterations to return the integer hull of $P_{n,\rho}$ for all non-integer $\rho \in (0, n-1)$.
- The integrality gap of $\mathrm{SA}_+^k(P_{n,\rho})$ in the direction of the all-ones vector is
$$1 + \frac{(n-k)(1-\rho)}{(n-1)(n-k+k\rho)}$$
for all $n \geq 2, k \in \{0,1,\ldots,n\}$, and $\rho \in (0,1)$. Moreover, we show that this integrality gap is exactly the same, if we replace SA_+ by an operator as weak as $\tilde{\mathrm{LS}}$.
- The Las-rank of $P_{n,\rho}$ is n for all $\rho \in \left(0, \frac{n^2-1}{2n^{n+1}-n^2-1}\right]$. This strengthens earlier work by Cheung [Che07], who showed the existence of such a positive ρ but did not give concrete bounds.
- The Las-rank of $Q_{n,\rho}$ is n for all $\rho \in \left(0, \frac{n+1}{2^{n+2}-n-3}\right)$, and at most $n-1$ for all $\rho > \frac{n}{2^{n+1}-2}$.
- There exist n, ρ where the BZ'_+-rank of $P_{n,\rho}$ is $\Omega(\sqrt{n})$, providing what we believe to be the first example where BZ'_+ (and as a consequence, the weaker BZ_+) requires more than a constant number of iterations to return the integer hull of a set.

The tools we use in our analysis, which involve zeta and moment matrices, build on earlier work by others (such as [Lau03] and [Che07]), and could be useful in analyzing lift-and-project relaxations of other sets. Finally, we conclude the manuscript by noting some interesting behaviour of the integrality gaps of some lift-and-project relaxations.

We remark that preliminary and weaker versions of our results on the Lasserre relaxations of $P_{n,\rho}$ and $Q_{n,\rho}$ were published in the first author's PhD thesis [Au14]. During the writing of this manuscript, we discovered that Kurpisz, Leppänen and Mastrolilli [KLM15] had obtained similar and stronger results. In fact, in their work, they characterized general conditions for when the $(n-1)^{\text{th}}$ Lasserre relaxation is not the integer hull. Using very similar ideas to theirs, we have subsequently sharpened our results to those appearing in this manuscript.

2. Preliminaries

In this section, we establish some notation and describe several lift-and-project operators utilizing positive semidefiniteness constraints.

2.1. The operators $\mathrm{SA}, \mathrm{SA}_+, \mathrm{SA}'_+$ and LS_+.

Let $\mathcal{F}$ denote $\{0,1\}^n$, and define $\mathcal{A} := 2^{\mathcal{F}}$, the power set of $\mathcal{F}$. As shown in [Zuc03], many existing lift-and-project operators can be seen as lifting a given relaxation P to a set of matrices whose rows and columns are indexed by sets in $\mathcal{A}$. For more motivation and details on this framework, the reader may refer to [AT16].

We first define the operator SA due to Sherali and Adams [SA90], while introducing the notation and notions that we shall build upon when we subsequently turn our focus to the more complicated operators. Given $P \subseteq [0,1]^n$, define the cone
$$K(P) := \left\{ \begin{pmatrix} \lambda \\ \lambda x \end{pmatrix} \in \mathbb{R}^{n+1} : \lambda \geq 0,\ x \in P \right\},$$
where we shall index the extra coordinate by 0. Next, we introduce a family of sets in $\mathcal{A}$ that are used extensively by the operators we will introduce in this paper. Given a set of indices $S \subseteq [n]$ and $t \in \{0,1\}$, we define
$$S|_t := \{x \in \mathcal{F} : x_i = t,\ \forall i \in S\}.$$

Note that $\emptyset|_0 = \emptyset|_1 = \mathcal{F}$. Also, to reduce cluttering, we write $i|_t$ instead of $\{i\}|_t$. Next, given any integer $\ell \in \{0, 1, \ldots, n\}$, we define

$$\mathcal{A}_\ell := \{S|_1 \cap T|_0 : S, T \subseteq [n], S \cap T = \emptyset, |S| + |T| \leq \ell\},$$

and

$$\mathcal{A}_\ell^+ := \{S|_1 : S \subseteq [n], |S| \leq \ell\}.$$

For instance, $\mathcal{A}_0 = \mathcal{A}_0^+ = \{\mathcal{F}\}$,

$$\mathcal{A}_1 = \{\mathcal{F}, 1|_1, 2|_1, \ldots, n|_1, 1|_0, 2|_0, \ldots, n|_0\},$$

and

$$\mathcal{A}_1^+ = \{\mathcal{F}, 1|_1, 2|_1, \ldots, n|_1\}.$$

Given a vector $y \in \mathbb{R}^{\mathcal{A}'}$ where $\mathcal{A}_1^+ \subseteq \mathcal{A}' \subseteq \mathcal{A}$, we let $\hat{x}(y) := (y_{\mathcal{F}}, y_{1|_1}, \ldots, y_{n|_1})^\top$. We usually use $\hat{x}(y)$ when we project y from a higher dimension to a vector in $\mathbb{R}^{n+1}$, and verify its membership in $K(P)$. Therefore, sometimes we may also alternatively index the entries of $\hat{x}(y)$ as $(y_0, y_1, \ldots, y_n)^\top$.

Finally, let e_i denote the unit vector (in the appropriate space) indexed by i (which could be an integer among $\{0, 1, \ldots, n\}$, or a set in $\mathcal{A}$; this will be made clear by the context). Then, given an integer $k \in [n]$, we define the operator SA^k as follows:

(1) Let $\widehat{\mathrm{SA}}^k(P)$ denote the set of matrices $Y \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_k}$ which satisfy all of the following conditions:
  - (SA 1) $Y[\mathcal{F}, \mathcal{F}] = 1$.
  - (SA 2) $Ye_\alpha \in K(P)$, for every $\alpha \in \mathcal{A}_k$.
  - (SA 3) For every $S|_1 \cap T|_0 \in \mathcal{A}_{k-1}$,
  $$Ye_{S|_1 \cap T|_0 \cap j|_1} + Ye_{S|_1 \cap T|_0 \cap j|_0} = Ye_{S|_1 \cap T|_0}, \quad \forall j \in [n] \setminus (S \cup T).$$
  - (SA 4) For all $\alpha \in \mathcal{A}_1^+, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta = \emptyset$, $Y[\alpha, \beta] = 0$.
  - (SA 5) For all $\alpha_1, \alpha_2 \in \mathcal{A}_1^+, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

(2) Define
$$\mathrm{SA}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\mathrm{SA}}^k(P), Ye_{\mathcal{F}} = \begin{pmatrix} 1 \\ x \end{pmatrix} \right\}.$$

Notice that we have presented the SA^k operator differently compared to some of its descriptions in the literature, as this broader framework (of lifting a set to matrices whose rows and columns are indexed by sets in $\mathcal{A}$) is needed when we later study the more complex operators (such as the Bienstock–Zuckerberg variants). To see the correspondence between the traditional description of the SA^k operator (in terms of linearizing polynomial inequalities of the input relaxation P) and ours, suppose we are given an inequality $\sum_{i=1}^n a_i x_i \leq a_0$ that is valid for P. Then if we take disjoint subsets of indices $S, T \subseteq [n]$ such that $|S| + |T| \leq k$, SA^k generates the inequality

$$\left(\prod_{i \in S} x_i\right)\left(\prod_{i \in T}(1 - x_i)\right)\left(\sum_{i=1}^n a_i x_i\right) \leq \left(\prod_{i \in S} x_i\right)\left(\prod_{i \in T}(1 - x_i)\right) a_0. \tag{1}$$

After expanding the products, each term of the form x_i^j (where $j \geq 2$) is replaced by x_i, and then each nontrivial product of monomials $\prod_{i \in U} x_i$ by a variable x_U to obtain a linear inequality. In our definition of SA^k, the corresponding linearized inequality is

$$\sum_{i=1}^{n} a_i Y[i|_1, S|_1 \cap T|_0] \leq a_0 Y[\mathcal{F}, S|_1 \cap T|_0],$$

which is imposed by (SA 2) on the column of Y indexed by the set $S|_1 \cap T|_0$. Also, in the original description of SA^k, the variable x_U is used to represent all appearances of the product $\prod_{i \in U} x_i$ in the formulation, which could arise when the inequality (1) is generated using multiple choices of S and T. In our description, this step of identifying all instances of the product $\prod_{i \in U} x_i$ with the same variable x_U is captured by (SA 5).

Observe that, in our description, the lifted space of SA^k is a set of matrices with $|\mathcal{A}_k| = \sum_{i=0}^{k} 2^i \binom{n}{i}$ columns. From the condition (SA 3), we see that many of these columns are linearly dependent. In fact, for every $Y \in \widehat{\mathrm{SA}}^k(P)$, the columns of Y corresponding to the sets in $\mathcal{A}_k^+$ always span the column space of Y (the same applies for the columns corresponding to sets in $\mathcal{A}_k \setminus \mathcal{A}_{k-1}$). Thus, one could define SA^k such that its lifted space consists of matrices with much fewer columns. We opted to include the "extra" columns because this way each "verify membership in $K(P)$" constraint imposed by (SA 2) only involves entries in a single matrix column in the lifted space, as opposed to linear combinations of up to 2^k columns had we only used columns representing sets in, say, $\mathcal{A}_k^+$.

Next, we define two strengthened variants of SA which we call SA_+ and SA'_+. Let $\mathbb{S}_+^n$ denote the set of n-by-n real, symmetric matrices that are positive semidefinite. Then, given a positive integer $k \in [n]$, we define the operators SA_+^k and $\mathrm{SA}_+'^k$ as follows:

(1) Let $\widehat{\mathrm{SA}}_+^k(P)$ be the set of matrices $Y \in \mathbb{S}_+^{\mathcal{A}_k}$ which satisfy all of the following conditions:

(SA_+ 1) $Y[\mathcal{F}, \mathcal{F}] = 1$.

(SA_+ 2) For every $\alpha \in \mathcal{A}_k$:

 (i) $\hat{x}(Ye_\alpha) \in K(P)$;

 (ii) $Ye_\alpha \geq 0$.

(SA_+ 3) For every $S|_1 \cap T|_0 \in \mathcal{A}_{k-1}$,

$$Ye_{S|_1 \cap T|_0 \cap j|_1} + Ye_{S|_1 \cap T|_0 \cap j|_0} = Ye_{S|_1 \cap T|_0}, \quad \forall j \in [n] \setminus (S \cup T).$$

(SA_+ 4) For all $\alpha, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta = \emptyset, Y[\alpha, \beta] = 0$.

(SA_+ 5) For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2, Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

(2) Let $\widehat{\mathrm{SA}}_+'^k(P)$ be the set of matrices in $\widehat{\mathrm{SA}}_+^k(P)$ that also satisfy:

(SA'_+ 4) For all $\alpha, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta \cap P = \emptyset, Y[\alpha, \beta] = 0$.

(3) Define

$$\mathrm{SA}_+^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\mathrm{SA}}_+^k(P), \hat{x}(Ye_{\mathcal{F}}) = \begin{pmatrix} 1 \\ x \end{pmatrix} \right\},$$

and

$$\mathrm{SA}_+'^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\mathrm{SA}}_+'^k(P), \hat{x}(Ye_{\mathcal{F}}) = \begin{pmatrix} 1 \\ x \end{pmatrix} \right\}.$$

The operators SA_+^k and $\mathrm{SA}_+'^k$ are rather similar in nature. First, SA_+^k is a strengthening of the SA^k operator. Not only does it impose all conditions required by SA^k, it extends the lifted space of SA^k (which consists of matrices of dimension $(n+1) \times \Theta(n^k)$) to a set of $\Theta(n^k)$-by-$\Theta(n^k)$ symmetric matrices, and imposes an additional positive semidefiniteness constraint on this large square matrix. Thus, given a set P and a matrix Y in $\widehat{\mathrm{SA}}_+^k(P)$, the submatrix of Y with the rows corresponding to sets in $\mathcal{A}_+^1$ would be in $\widehat{\mathrm{SA}}^k(P)$. Hence, it follows that $\mathrm{SA}_+^k(P) \subseteq \mathrm{SA}^k(P)$, for every set $P \subseteq [0,1]^n$ and for every $k \geq 1$.

Next, to see that $\mathrm{SA}_+^k(P)$ is indeed a relaxation of P_I, observe that given any $x \in P \cap \{0,1\}^n$, if we define the vector $x' \in \mathbb{R}^{\mathcal{A}_k}$ where

$$x'[S|_1 \cap T|_0] = \begin{cases} 1 & \text{if } x_i = 1, \ \forall i \in S \text{ and } x_i = 0, \ \forall i \in T; \\ 0 & \text{otherwise,} \end{cases}$$

then it is easy to check that the matrix $x'(x')^\top$ is in $\widehat{\mathrm{SA}}_+^k(P)$. Since $\mathrm{SA}_+^k(P)$ is a convex set by construction, it follows that $P_I \subseteq \mathrm{SA}_+^k(P)$.

As for $\mathrm{SA}_+'^k$, observe that the condition $(\mathrm{SA}_+'\,4)$ is more restrictive than $(\mathrm{SA}_+\,4)$, and thus $\mathrm{SA}_+'^k$ forces more variables to be zero in the lifted space, and potentially produces a tighter relaxation. Thus, $\mathrm{SA}_+'^k(P) \subseteq \mathrm{SA}_+^k(P)$. Conversely, it was shown in [Au14] that $\mathrm{SA}_+^{2k}(P) \subseteq \mathrm{SA}_+'^k(P)$ in general (i.e. strengthening SA_+ by imposing the additional requirement $(\mathrm{SA}_+'\,4)$ can only reduce the lift-and-project rank of any set by a factor of at most 2). We will mostly focus on SA_+, and only use SA_+' for relating the performance of SA_+ and the Bienstock–Zuckerberg operators in certain situations.

Figure 1 illustrates the structure of the lifted spaces of SA_+^k and $\mathrm{SA}_+'^k$ compared to SA^k. As with SA^k, our description of SA_+^k and $\mathrm{SA}_+'^k$ ensures that the matrices in their lifted spaces have many linear dependencies among their rows and columns, as this will make our subsequent analyses of these operators simpler. Similar but weaker versions of SA_+^k have been considered in the literature, such as the *Sherali–Adams SDP* operator studied in [CS08] and [BGMT12], whose positive semidefiniteness constraint is only applied on the $\mathcal{A}_1^+ \times \mathcal{A}_1^+$ symmetric minor for all $k \geq 1$. In contrast, our version of SA_+^k requires the entire $\mathcal{A}_k \times \mathcal{A}_k$ matrix displayed in Figure 1 to be positive semidefinite.

Finally, LS_+, the operator defined in [LS91] that utilizes positive semidefiniteness, is equivalent to SA_+^1. Then if we let LS_+^k denote k iterative applications of LS_+, it was shown in [AT16] that SA_+^k dominates LS_+^k in general. That is, for every set $P \in [0,1]^n$, $P_I \subseteq \mathrm{SA}_+^k(P) \subseteq \mathrm{LS}_+^k(P)$.

2.2. **The Lasserre operator.** We now turn our attention to the Las operator due to Lasserre [Las01]. While Las can be applied to semialgebraic sets, we restrict our discussion to its applications to polytopes contained in $[0,1]^n$. Gouveia, Parrilo and Thomas provided in [GPT10] an alternative description of the Las operator, where P_I is described as the variety of an ideal intersected with the solutions to a system of polynomial inequalities. Our presentation of the operator is closer to that in [Lau03] than to Lasserre's original description. Given $P := \{x \in [0,1]^n : Ax \leq b\}$ (where A is an m-by-n matrix and $b \in \mathbb{R}^m$), and an integer $k \in [n]$,

(1) Let $\widehat{\mathrm{Las}}^k(P)$ denote the set of matrices $Y \in \mathbb{S}_+^{\mathcal{A}_{k+1}^+}$ that satisfy all of the following conditions:

(Las 1) $Y[\mathcal{F}, \mathcal{F}] = 1$;

(Las 2) For every $i \in [m]$, define the matrix $Y^i \in \mathbb{S}^{\mathcal{A}_k^+}$ where

$$Y^i[S|_1, S'|_1] := b_i Y[S|_1, S'|_1] - \sum_{j=1}^n A[i,j] Y[(S \cup \{j\})|_1, (S' \cup \{j\})|_1],$$

and impose $Y^i \succeq 0$.

(Las 3) For every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k^+$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

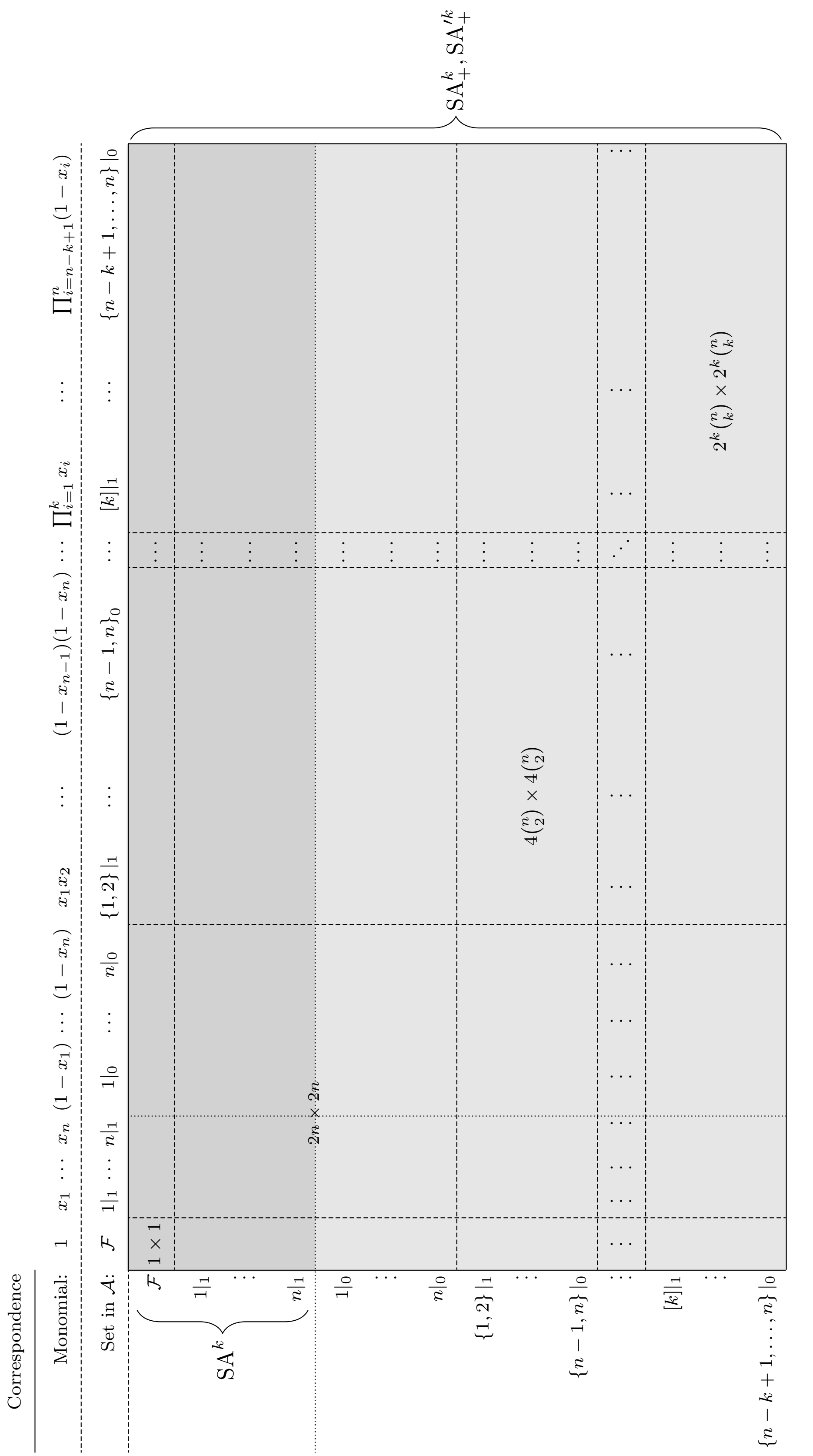

Figure 1. Illustrating the lifted spaces of the Sherali-Adams based operators

(2) Define

$$\mathrm{Las}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\mathrm{Las}}^k(P) : \hat{x}(Ye_{\mathcal{F}}) = \begin{pmatrix} 1 \\ x \end{pmatrix} \right\}.$$

For all operators Γ considered in this paper, and for every polytope $P \subseteq [0,1]^n$, we define $\Gamma^0(P) := P$.

We note that, unlike the previously mentioned operators, Las requires an explicit description of P in terms of valid inequalities. While it is not apparent in the above definition of the Las operator (as it only uses the variables in the form $S|_1$, instead of the broader family of $S|_1 \cap T|_0$ as in operators based on SA), we show that Las does commute with all automorphisms of the unit hypercube.

Proposition 1. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation such that $\{L(x) : x \in [0,1]^n\} = [0,1]^n$. Then, $\mathrm{Las}^k(L(P)) = L(\mathrm{Las}^k(P))$ for all polytopes $P \subseteq [0,1]$ and for every positive integer k.*

Proof. Since the automorphism group of the unit hypercube is generated by linear transformations swapping two coordinates and affine transformations flipping a coordinate, it suffices to prove that Las commutes with each of these transformations. First, we show that Las^k commutes with the mappings which swap two coordinates. Without loss of generality, we may assume the coordinates are 1 and 2. Let L_1 denote the linear transformation, where

$$[L_1(x)]_i := \begin{cases} x_2 & \text{if } i = 1; \\ x_1 & \text{if } i = 2; \\ x_i & \text{otherwise.} \end{cases}$$

We also define the map $\mathcal{L} : \mathcal{A}_{k+1}^+ \to \mathcal{A}_{k+1}^+$ where

$$\mathcal{L}(S|_1) := \begin{cases} ((S \setminus \{1\}) \cup \{2\})|_1 & \text{if } 1 \in S, 2 \notin S; \\ ((S \setminus \{2\}) \cup \{1\})|_1 & \text{if } 2 \in S, 1 \notin S; \\ S|_1 & \text{otherwise,} \end{cases}$$

Now suppose $x \in \mathrm{Las}^k(P)$, with certificate matrix $Y \in \widehat{\mathrm{Las}}^k(P)$. We show that $L_1(x) \in \mathrm{Las}^k(L_1(P))$. Define $Y' \in \mathbb{S}^{\mathcal{A}_{k+1}^+}$ such that

$$Y'[S|_1, T|_1] := Y[\mathcal{L}(S)|_1, \mathcal{L}(T)|_1], \quad \text{for all } S, T \in \mathcal{A}_{k+1}.$$

Then we see that Y' is Y with some columns and rows permuted, and thus is positive semidefinite too. Next, for each $a \in \mathbb{R}^{n+1}$ such that $a_0 + \sum_{i=1}^n a_i x_i \geq 0$ is an inequality in the system describing P, define $a' \in \mathbb{R}^{n+1}$ where

$$a'_i := \begin{cases} a_2 & \text{if } i = 1; \\ a_1 & \text{if } i = 2; \\ a_i & \text{otherwise.} \end{cases}$$

Then the collection of the derived inequalities $a'_0 + \sum_{i=1}^{n} a'_i x_i \geq 0$ describe $L(P)$. If this is the j^{th} inequality describing $L(P)$, then

$$\begin{aligned}
& Y'^j[S|_1, T|_1] \\
= \;& a'_0 Y'[S|_1, T|_1] + \sum_{i=1}^{n} a'_i Y'[(S \cup \{i\})|_1, (T \cup \{i\})|_1] \\
= \;& a_0 Y[\mathcal{L}(S)|_1, \mathcal{L}(T)|_1] + a_2 Y[\mathcal{L}(S \cup \{1\})|_1, \mathcal{L}(T \cup \{1\})|_1] + a_1 Y[\mathcal{L}(S \cup \{2\})|_1, \mathcal{L}(T \cup \{2\})|_1] \\
& + \sum_{i=3}^{n} Y[\mathcal{L}(S \cup \{i\})|_1, \mathcal{L}(T \cup \{i\}))|_1] \\
= \;& a_0 Y[\mathcal{L}(S)|_1, \mathcal{L}(T)|_1] + a_2 Y[\mathcal{L}(S) \cup \{2\}\,|_1, \mathcal{L}(T) \cup \{2\}\,|_1] + a_1 Y[\mathcal{L}(S) \cup \{1\}\,|_1, \mathcal{L}(T) \cup \{1\}\,|_1] \\
& + \sum_{i=3}^{n} Y[\mathcal{L}(S \cup \{i\})|_1, \mathcal{L}(T \cup \{i\}))|_1] \\
= \;& Y^j[\mathcal{L}(S)|_1, \mathcal{L}(T)|_1].
\end{aligned}$$

Thus, Y'^j is also Y^j with rows and columns permuted, and thus is positive semidefinite. Hence, we obtain that $\hat{x}(Y' e_{\mathcal{F}}) = L_1(x)$ is in $\text{Las}(L_1(P))$.

Next, consider the affine transformations flipping a coordinate (without loss of generality, the first coordinate). So, we define $L_2 : \mathbb{R}^n \to \mathbb{R}^n$ where

$$[L_2(x)]_i := \begin{cases} 1 - x_1 & \text{if } i = 1; \\ x_i & \text{otherwise.} \end{cases}$$

Also, for every integer $\ell \geq 1$, define $U^{(\ell)} \in \mathbb{R}^{\mathcal{A}_\ell^+ \times \mathcal{A}_\ell}$ such that

$$U^{(\ell)}[S|_1, T|_1 \cap W|_0] := \begin{cases} (-1)^{|S \setminus T|} & \text{if } T \subseteq S \subseteq T \cup W; \\ 0 & \text{otherwise.} \end{cases}$$

Now let $x \in \text{Las}^k(P)$, with certificate matrix $Y \in \widehat{\text{Las}}^k(P)$. Define $\bar{Y} \in \mathbb{S}^{\mathcal{A}_{k+1}}$ where $\bar{Y} := (U^{(k+1)})^\top Y U^{(k+1)}$. This time, we let $\mathcal{L} : \mathcal{A}_{k+1}^+ \to \mathcal{A}_{k+1}$ denote the map where

$$\mathcal{L}(S|_1) := \begin{cases} ((S \setminus \{1\})|_1 \cap 1|_0 & \text{if } 1 \in S; \\ S|_1 & \text{otherwise,} \end{cases}$$

and let $Y' \in \mathbb{S}^{\mathcal{A}_{k+1}^+}$ such that

$$Y'[S|_1, T|_1] := \bar{Y}[\mathcal{L}(S)|_1, \mathcal{L}(T)|_1], \quad \text{for all } S, T \in \mathcal{A}_{k+1}.$$

Then we see that Y' is a symmetric minor of $\bar{Y} = (U^{(k+1)})^\top Y U^{(k+1)}$. Since $Y \succeq 0$, it follows that $Y' \succeq 0$ as well. Next, for each $a \in \mathbb{R}^{n+1}$ such that $a_0 + \sum_{i=1}^{n} a_i x_i \geq 0$ is an inequality in the system describing P, define $a' \in \mathbb{R}^{n+1}$ where

$$a'_i := \begin{cases} a_0 + a_1 & \text{if } i = 0; \\ -a_1 & \text{if } i = 1; \\ a_i & \text{otherwise.} \end{cases}$$

Then the collection of the derived inequalities $a'_0 + \sum_{i=1}^{n} a'_i x_i \geq 0$ describe $L(P)$. If this is the j^{th} inequality describing $L(P)$, then

$$\begin{aligned}
& Y'^j[S|_1, T|_1] \\
= \;& a'_0 Y'[S|_1, T|_1] + \sum_{i=1}^{n} a'_i Y'[(S \cup \{i\})|_1, (T \cup \{i\})|_1] \\
= \;& (a_0 + a_1)\bar{Y}[\mathcal{L}(S)|_1, \mathcal{L}(T)|_1] - a_1 \bar{Y}[\mathcal{L}(S \cup \{1\})|_1, \mathcal{L}(T \cup \{1\})|_1] \\
& + \sum_{i=2}^{n} \bar{Y}[\mathcal{L}(S \cup \{i\})|_1, \mathcal{L}(T \cup \{i\})|_1] \\
= \;& a_0 \bar{Y}[\mathcal{L}(S)|_1, \mathcal{L}(T)|_1] + a_1 \bar{Y}[\mathcal{L}(S) \cup \{1\}\,|_1, \mathcal{L}(T) \cup \{1\}\,|_1] \\
& + \sum_{i=2}^{n} \bar{Y}[\mathcal{L}(S) \cup \{i\}\,|_1, \mathcal{L}(T) \cup \{i\}\,|_1] \\
= \;& ((U^{(k)})^\top Y^j U^{(k)})[\mathcal{L}(S)|_1, \mathcal{L}(T)|_1].
\end{aligned}$$

Thus, Y'^j is a symmetric minor of $(U^{(k)})^\top Y^j U^{(k)}$, and thus is positive semidefinite. Therefore, $\hat{x}(Y' e_{\mathcal{F}}) = L_2(x)$ is in $\mathrm{Las}(L_2(P))$. □

2.3. **The Bienstock–Zuckerberg operator.** In [BZ04], Bienstock and Zuckerberg devised a positive semidefinite lift-and-project operator (which we denote BZ_+ herein) that is quite different from the previously (pre-2004) proposed operators. In particular, in its lifted space, it utilizes variables in $\mathcal{A}$ that are not necessarily in the form $S|_1 \cap T|_0$, in addition to a number of other ideas. One such idea is *refinement*. While BZ_+ is defined for any polytope contained in $[0,1]^n$, we will restrict our discussion to lower-comprehensive polytopes for simplicity's sake. A polytope P is defined to be *lower-comprehensive* if, given any $x \in P$, every vector y where $0 \leq y \leq x$ is also in P. Note that many natural relaxations of packing-type problems (such as the stable set problem and maximum matching problem of graphs) are lower-comprehensive.

Let polytope $P := \{x \in [0,1]^n : Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}$ is nonnegative and $b \in \mathbb{R}^m$ is positive (this implies that P is lower-comprehensive; conversely, every n-dimensional lower-comprehensive polytope in $[0,1]^n$ admits such a representation). Next, given a vector v, let $\mathrm{supp}(v)$ denote the *support* of v. Also, we define a subset O of $[n]$ to be a *k-small obstruction* of P if there exists an inequality $a^\top x \leq b_i$ in the system $Ax \leq b$ where

- $O \subseteq \mathrm{supp}(a)$;
- $\sum_{j \in O} a_j > b_i$; and
- $|O| \leq k+1$ or $|O| \geq |\mathrm{supp}(a)| - (k+1)$.

Observe that, given such an obstruction O, the inequality $\sum_{i \in O} x_i \leq |O| - 1$ holds for every integral vector $x \in P$. Thus, if we let $\mathcal{O}_k$ denote the collection of all k-small obstructions of the system $Ax \leq b$, then the set

$$\mathcal{O}_k(P) := \left\{ x \in P : \sum_{i \in O} x_i \leq |O| - 1, \ \forall O \in \mathcal{O}_k \right\}$$

is a relaxation of P_I that is potentially tighter than P. After tightening P with these obstruction inequalities, BZ_+ lifts $\mathcal{O}_k(P)$ to a higher space, whose dimensions depend on collections of indices called walls and tiers. Define

$$\mathcal{W}_k := \left\{ \bigcup_{i,j \in [\ell], i \neq j} (O_i \cap O_j) : O_1, \ldots, O_\ell \in \mathcal{O}_k, \ell \leq k+1 \right\} \cup \{\{1\}, \ldots, \{n\}\}$$

to be the collection of *walls*. That is, each wall is either a singleton set of an index in $[n]$, or is generated by a subset of up to $(k+1)$ k-small obstructions, where a wall is formed by the set of elements that appear in at least two of the given obstructions. Next, we define the collection of *tiers*

$$\mathcal{T}_k := \left\{ S \subseteq [n] : \exists W_{i_1}, \ldots, W_{i_k} \in \mathcal{W}_k, S \subseteq \bigcup_{j=1}^{k} W_{i_j} \right\}.$$

That is, a set of indices S is a tier if it is contained in the union of a set of up to k walls. Observe that since all singleton sets are walls, every subset of $[n]$ of size at most k is a tier.

Next, we present the details of BZ'_+, a simplified variant of BZ_+ that is shown in [AT16] to dominate BZ_+. Intuitively, BZ'_+ is less selective than BZ_+ in generating variables, and could lift P to a lifted space whose dimensions is exponential in n. Since we shall only use BZ'_+ to establish hardness results, the fact that it may not produce tractable relaxations is not a concern here.

Also, observe that the collections of obstructions, walls and tiers depend on the algebraic description of a set. Thus, BZ_+ and BZ'_+ could lift two sets in $[0,1]^n$ (or even two different systems of inequalities describing the same set) to lifted spaces of different dimensions. In contrast, the dimensions of the lifted space for all other operators we introduced earlier only depend on the dimension of the input set P. Bienstock and Zuckerberg showed in [BZ04] that the adaptivity enables BZ_+ to efficiently solve some set-covering type problem instances that could require exponential effort by the earlier operators.

On the other hand, one can construct instances where the system $Ax \leq b$ describing the input set P does not have a single k-small obstruction. In that case, the only walls generated are the singleton sets, and the tiers are exactly the subsets of indices of size at most k. This is one of the main ideas we will utilize in showing that BZ'_+ performs poorly on certain families of chipped hypercubes. While we present the full details of BZ'_+ below for completeness, the reader should be aware that many of these details will not be triggered in our subsequent analysis of BZ'_+.

While variants based on the SA operator only generate variables of the form $S|_1 \cap T|_0$ in the lifted space, variables in the BZ'_+ formulation can take a more general form. Given a set $U \subseteq [n]$ and a nonnegative integer r, we define

$$U|_{<r} := \left\{ x \in \mathcal{F} : \sum_{i \in U} x_i \leq r-1 \right\}.$$

Then all variables generated by BZ'_+ in the lifted space correspond to $S|_1 \cap T|_0 \cap U|_{<r}$ for disjoint sets of indices S, T and U. Next, given an integer $k \in [n]$, the BZ'_+ operator can be described as follows:

(1) Define $\mathcal{A}'$ to be the set consisting of the following. For each tier $S \in \mathcal{T}_k$, include:

$$(S \setminus T)|_1 \cap T|_0, \tag{2}$$

for all $T \subseteq S$ such that $|T| \leq k$; and

$$(S \setminus (T \cup U))|_1 \cap T|_0 \cap U|_{<|U|-(k-|T|)}, \tag{3}$$

for every $T, U \subseteq S$ such that $U \cap T = \emptyset$, $|T| < k$ and $|U| + |T| > k$. We say these variables (indexed by the above sets) are associated with the tier S.

(2) Let $\widehat{\mathrm{BZ}}'^{k}_{+}(P)$ denote the set of matrices $Y \in \mathbb{S}_{+}^{\mathcal{A}'}$ that satisfy all of the following conditions:

(BZ′ 1) $Y[\mathcal{F}, \mathcal{F}] = 1$.

(BZ′ 2) For every column y of the matrix Y,

(i) $0 \le y_\alpha \le y_{\mathcal{F}}$, for all $\alpha \in \mathcal{A}'$.

(ii) $\hat{x}(y) \in K(\mathcal{O}_k(P))$.

(iii) $y_{i|_1} + y_{i|_0} = y_{\mathcal{F}}$, for every $i \in [n]$.

(iv) For each $\alpha \in \mathcal{A}'$ in the form of $S|_1 \cap T|_0$ impose the inequalities

$$y_{i|_1} \ \ge \ y_\alpha, \quad \forall i \in S; \tag{4}$$

$$y_{i|_0} \ \ge \ y_\alpha, \quad \forall i \in T; \tag{5}$$

$$y_\alpha + y_{(S \cup \{i\})|_1 \cap (T \setminus \{i\})|_0} \ = \ y_{S|_1 \cap (T \setminus \{i\})|_0}, \quad \forall i \in T; \tag{6}$$

$$\sum_{i \in S} y_{i|_1} + \sum_{i \in T} y_{i|_0} - y_\alpha \ \le \ (|S| + |T| - 1) y_{\mathcal{F}}. \tag{7}$$

(v) For each $\alpha \in \mathcal{A}'$ in the form $S|_1 \cap T|_0 \cap U|_{<r}$, impose the inequalities

$$y_{i|_1} \ \ge \ y_\alpha, \quad \forall i \in S; \tag{8}$$

$$y_{i|_0} \ \ge \ y_\alpha, \quad \forall i \in T; \tag{9}$$

$$\sum_{i \in U} y_{i|_0} \ \ge \ (|U| - (r-1)) y_\alpha; \tag{10}$$

$$y_\alpha \ = \ y_{S|_1 \cap T|_0} - \sum_{U' \subseteq U, |U'| \ge r} y_{(S \cup U')|_1 \cap (T \cup (U \setminus U'))|_0}. \tag{11}$$

(BZ′ 3) For all $\alpha, \beta \in \mathcal{A}'$ such that $\alpha \cap \beta \cap P = \emptyset$, $Y[\alpha, \beta] = 0$.

(BZ′ 4) For all $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathcal{A}'$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

(3) Define

$$\mathrm{BZ}'^{k}_{+}(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\mathrm{BZ}}'^{k}_{+}(P), \hat{x}(Ye_{\mathcal{F}}) = \begin{pmatrix} 1 \\ x \end{pmatrix} \right\}.$$

Again, observe that in the case when the inequalities describing P does not have any k-small obstructions, every tier S has size at most k. Then all variables generated by BZ'_+ would be through (2), and hence are of the form $S|_1 \cap T|_0$. With no variables generated through (3), many subsequent conditions (such as (10) and (11)) are not triggered, and matrices in $\widehat{\mathrm{BZ}}'^{k}_{+}(P)$ have dimensions $\mathcal{A}_k \times \mathcal{A}_k$. In this case, the performance of BZ'_+ is comparable to that of SA_+:

Proposition 2. *If $P = \{x \in [0,1]^n : Ax \le b\}$ where $Ax \le b$ does not have a single k-small obstruction, then*

$$\mathrm{SA}^{2k}_{+}(P) \subseteq \mathrm{BZ}'^{k}_{+}(P).$$

Proof. If P does not have a single obstruction, then $\mathcal{O}_k(P) = P$, every wall is a singleton set and every tier has size at most k. Since there are no tiers of size greater than k, it is vacuously true that every tier of size greater than k is P-useless (this concept of *P-useless* is defined in [AT16]), and thus by Proposition 4 in [AT16], we obtain that $\mathrm{SA}'^{k}_{+}(P) \subseteq \mathrm{BZ}'^{k}_{+}(P)$. Since $\mathrm{SA}^{2k}_{+}(P) \subseteq \mathrm{SA}'^{k}_{+}(P)$ in general, our claim follows. □

In Figure 2 we provide a comparison of relative strengths of all aforementioned lift-and-project operators, in addition to BCC, a simple operator defined by Balas, Ceria, and Cornuéjols in [BCC93]; and $\tilde{\mathrm{LS}}$, a geometric operator studied in [GT01] in their analysis of the Lovász–Schrijver operators. Each arrow in the figure denotes "is dominated by", meaning that when applied to the same relaxation P, the operator at the head of an arrow would return a relaxation that is at least as tight as that obtained by applying the operator at the tail of the arrow. While the focus in this paper will be on the performance of $\mathrm{SA}_+, \mathrm{Las}$ and BZ'_+, some of our results also have implications on these other operators. The reader may refer to [AT16] for some more intricate properties of these operators.

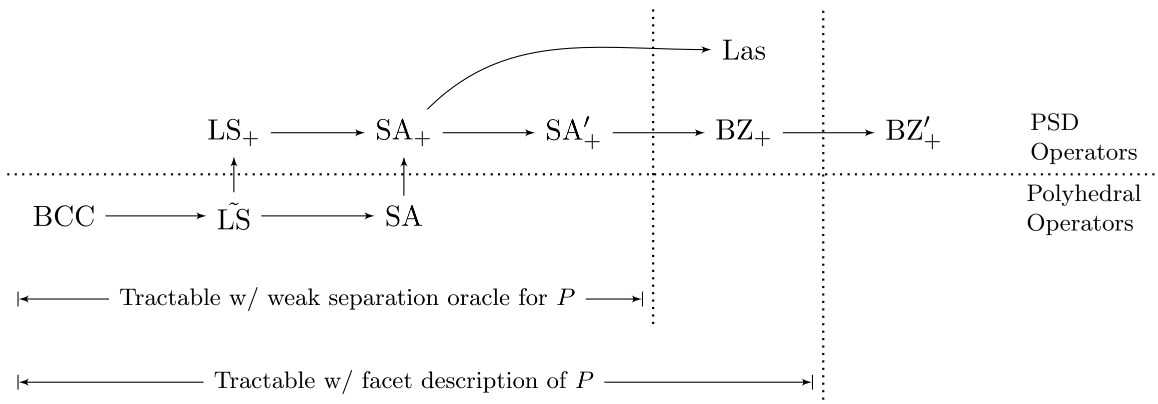

FIGURE 2. A strength chart of some lift-and-project operators.

There are also many other operators whose relative performance can be studied in this wider context of operators. For example, recently Bodur, Dash and Günlük [BDG16] proposed a polyhedral lift-and-project operator called $\tilde{\mathrm{N}}$ and showed that

$$\mathrm{LS} \to \tilde{\mathrm{N}} \to \mathrm{SA}^2$$

where LS is a polyhedral operator devised in [LS91] that dominates $\tilde{\mathrm{LS}}$.

Considering Figure 2, note that every lower bound that we prove on rank as well as integrality gaps for Las and BZ'_+ imply the same results for all other operators in Figure 2. Similarly, every upper bound on rank and integrality gaps for $\tilde{\mathrm{LS}}$ applies to all other operators in Figure 2, except BCC.

3. Some bad instances for $\mathrm{SA}_+, \mathrm{Las}$ and BZ'_+

In this section, we consider several polytopes that have been shown to be bad instances for many known lift-and-project operators (and cutting plane schemes in general).

3.1. The chipped hypercube $P_{n,\rho}$.

Recall the chipped hypercube

$$P_{n,\rho} := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq n - \rho \right\}.$$

Cook and Dash [CD01] showed that the LS_+-rank of $P_{n,1/2}$ is n, while Laurent [Lau03] proved that the SA-rank of $P_{n,1/2}$ is also n. Cheung [Che07] extended these results and showed that both the LS_+- and SA-rank of $P_{n,\rho}$ are n for all $\rho \in (0,1)$. Here, we use similar techniques to establish the SA_+-rank for $P_{n,\rho}$. Note that, from here on, we will sometimes use $v[i]$ to denote the i-entry of a vector v (instead of v_i).

Proposition 3. *For every $n \geq 2$, the SA_+-rank of $P_{n,\rho}$ is n for all $\rho \in (0,1)$.*

Proof. We prove our claim by showing that $\bar{x} := \left(1 - \frac{\rho}{n\rho+1-\rho}\right)\bar{e} \in \operatorname{SA}_+^{n-1}(P_{n,\rho}) \setminus (P_{n,\rho})_I$, where $\bar{e}$ denotes the all-ones vector. First,

$$\sum_{i=1}^{n} \bar{x}_i = n\left(1 - \frac{\rho}{n\rho+1-\rho}\right) = n - \frac{n\rho}{n\rho+1-\rho} > n-1,$$

and so $\bar{x} \notin (P_{n,\rho})_I$. We next show that this vector is in $\operatorname{SA}_+^{n-1}(P_{n,\rho})$. Define $Y \in \mathbb{R}^{\mathcal{A}_{n-1} \times \mathcal{A}_{n-1}}$ such that

$$Y[\alpha, \beta] := \begin{cases} 1 - \frac{\rho|S|}{n\rho+1-\rho} & \text{if } \alpha \cap \beta = S|_1 \text{ for some } S \subseteq [n]; \\ \frac{\rho}{n\rho+1-\rho} & \text{if } \alpha \cap \beta = S|_1 \cap j|_0 \text{ for some } S \subseteq [n] \text{ and } j \in [n] \setminus S; \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $Y \in \widehat{\operatorname{SA}}_+^{n-1}(P_{n,\rho})$. First, ($\operatorname{SA}_+$ 1) holds as $Y[\mathcal{F}, \mathcal{F}] = 1$ (since $\mathcal{F} \cap \mathcal{F} = \emptyset|_1$). It is not hard to see that $Y \geq 0$, as every entry in Y is either 0, $\frac{\rho}{n\rho+1-\rho}$ or $1 - \frac{k\rho}{n\rho+1-\rho}$ for some integer $k \in \{0, \ldots, n\}$. Next, we check that $\hat{x}(Ye_\beta) = (Y[\mathcal{F}, \beta], Y[1|_1, \beta], \ldots, Y[n|_1, \beta])^\top \in K(P_{n,\rho})$ for all $\beta \in \mathcal{A}_{n-1}$. Given $\beta = S|_1 \cap T|_0$, $\hat{x}(Ye_\beta)$ is the zero vector whenever $|T| \geq 2$, and is the vector $\frac{\rho}{n\rho+1-\rho}(\bar{e} - e_i)$ whenever $T = \{i\}$ for some $i \in [n]$.

Finally, suppose $\beta = S|_1$ for some $S \subseteq [n]$ where $|S| = k$. Then

$$\hat{x}(Ye_\beta)[i] = \begin{cases} 1 - \frac{k\rho}{n\rho+1-\rho} & \text{if } i = 0 \text{ or } i \in S; \\ 1 - \frac{(k+1)\rho}{n\rho+1-\rho} & \text{if } i \in [n] \setminus S. \end{cases}$$

Now

$$\begin{aligned} \sum_{i=1}^{n} \hat{x}(Ye_\beta)[i] &= k\left(1 - \frac{k\rho}{n\rho+1-\rho}\right) + (n-k)\left(1 - \frac{(k+1)\rho}{n\rho+1-\rho}\right) \\ &= n\left(1 - \frac{k\rho}{n\rho+1-\rho}\right) - \rho\left(\frac{n-k}{n\rho+1-\rho}\right) \\ &\leq (n-\rho)\left(1 - \frac{k\rho}{n\rho+1-\rho}\right) \\ &= (n-\rho)\hat{x}(Ye_\beta)[0]. \end{aligned}$$

Thus, $\hat{x}(Ye_\beta) \in K(P)$ in this case as well. Next, it is not hard to see that the entries of Y satisfy (SA_+ 3), (SA_+ 4) and (SA_+ 5). Finally, to see that $Y \succeq 0$, let Y' be the symmetric minor of Y indexed by rows and columns from $\mathcal{A}_{n-1}^- := \{S|_0 : S \subseteq [n], |S| \leq n-1\}$. Then $Y' \succeq 0$ as it is diagonally dominant. Next, define $L \in \mathbb{R}^{\mathcal{A}_{n-1} \times \mathcal{A}_{n-1}^-}$ where

$$L[S|_1 \cap T|_0, U|_0] := \begin{cases} (-1)^{|S|} & \text{if } S \cup T = U; \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be checked that $Y = LY'L^\top$. Hence, we conclude that $Y \succeq 0$ as well. This completes our proof. $\square$

We next show that $(0,1)$ is the only range of ρ's for which the SA_+-rank of $P_{n,\rho}$ is n. To do that, it is helpful to introduce the notion of *moment matrices*. Given an integer $k \geq 0$ and vector $y \in \mathbb{R}^{\mathcal{A}_\ell^+}$ where $\ell \geq \min\{n, 2k\}$, we define the matrix $\mathcal{M}_k(y) \in \mathbb{R}^{\mathcal{A}_k^+ \times \mathcal{A}_k^+}$ where $\mathcal{M}_k(y)[\alpha, \beta] := y[\alpha \cap \beta]$ for all $\alpha, \beta \in \mathcal{A}_k^+$.

We also need the notion of ℓ-establishment, which was introduced in [AT16] and utilizes the presence of a certain set of variables in the lifted space, as well as a positive semidefiniteness constraint, to provide a guarantee on the overall performance of the operator. Suppose $Y \in \mathbb{S}^{\mathcal{A}'}$ for some $\mathcal{A}' \subseteq \mathcal{A}$. We say that Y is *ℓ-established* if all of the following conditions hold:

(ℓ1) $Y[\mathcal{F},\mathcal{F}] = 1$.
(ℓ2) $Y \succeq 0$.
(ℓ3) $\mathcal{A}_\ell^+ \subseteq \mathcal{A}'$.
(ℓ4) For all $\alpha, \beta, \alpha', \beta' \in \mathcal{A}_\ell^+$ such that $\alpha \cap \beta = \alpha' \cap \beta', Y[\alpha,\beta] = Y[\alpha',\beta']$.
(ℓ5) For all $\alpha, \beta \in \mathcal{A}_\ell^+, Y[\mathcal{F},\beta] \geq Y[\alpha,\beta]$.

It follows immediately from the definition of SA_+ that all matrices in $\widehat{\mathrm{SA}}_+^k(P)$ are k-established, for all $P \subseteq [0,1]^n$. Then we have the following:

Proposition 4. *For every $n \geq 2$ and non-integer $\rho \in (0,n)$, the SA_+-rank of $P_{n,\rho}$ is at most $n - \lceil \rho \rceil + 1$.*

Proof. Let $P := P_{n,\rho}$. First, let $\ell := n - \lceil \rho \rceil$. Observe that every matrix $Y \in \widehat{\mathrm{SA}}_+^{\ell+1}(P)$ is $(\ell+1)$-established (and thus ℓ-established), and the condition (SA_+ 5) guarantees that the symmetric minor of Y with rows and columns indexed by sets in $\mathcal{A}_\ell^+$ is a moment matrix $\mathcal{M}_\ell(y)$ for some vector y.

Now notice that for every set $S \subseteq [n]$ where $|S| = \ell + 1$, the incidence vector of S is not in P (as $\ell + 1 > n - \rho$). Hence, the condition $\hat{x}\left(Ye_{S|_1}\right) \in K(P)$ imposed by (SA_+ 2) implies that $Y[\mathcal{F}, S|_1] = 0$. Thus, we obtain that $Y[S|_1, S|_1] = 0$ for all $S \subseteq [n]$ of size $\ell + 1$, and so the diagonal entries of the symmetric minor Y' of Y indexed by sets in $\mathcal{A}_{\ell+1}^+ \setminus \mathcal{A}_\ell$ are all zero. For Y to be positive semidefinite, Y' must have all zero entries. Thus, we obtain that $y[S|_1] = 0$ for all sets S where $|S| \geq \ell + 1$.

Next, if we define $Z_i := \sum_{S \subseteq [n], |S| = i} y[S|_1]$ for every $i \geq 0$, we obtain that $Z_i = 0$ for all $i > \ell$. Then it follows from Corollary 12 in [AT16] that $Z_1 \leq \ell$. Since

$$Z_1 = \sum_{i=1}^{n} y[i|_1] = \sum_{i=1}^{n} Y[i|_1, \mathcal{F}],$$

we conclude that $\sum_{i=1}^n x_i \leq \ell$ is valid for $\mathrm{SA}_+^{\ell+1}(P)$, and our claim follows. $\square$

Thus, we know that the SA_+-rank of $P_{n,\rho}$ is exactly n when $\rho \in (0,1)$, and the rank is 1 if $\rho \in (n-1,n)$. When $\rho \in (n-2,n-1)$, it follows from Proposition 4 that the SA_+-rank is at most 2. Since it is not hard to show that $\mathrm{SA}_+^1(P_{n,\rho}) \neq P_{n,n-1}$, we know in this case that the SA_+-rank is exactly 2.

Next, we show that for a weaker operator, the rank of $P_{n,\rho}$ is always n if it is not integral and strictly contains the unit simplex. Given integer $k \in [n]$ and $P \subseteq [0,1]^n$, consider the following operator originally defined in [GT01]:

$$\tilde{\mathrm{LS}}^k(P) := \bigcap_{S \subseteq [n], |S| = k} \operatorname{conv}\left\{x \in P : x_i \in \{0,1\},\ \forall i \in S\right\}.$$

That is, x is in $\tilde{\mathrm{LS}}^k(P)$ if and only if for every set of indices S of size k, x can be expressed as a convex combination of points in P whose entries in S are all integral. While $\tilde{\mathrm{LS}}$ produces tighter relaxations than BCC, it in turn is dominated by SA and several operators devised by Lovász and Schrijver in [LS91] (see, for instance, [GT01] for a discussion on this matter). Then we have the following:

Proposition 5. *For every integer $n \geq 2$ and for every non-integer $\rho \in (0, n-1)$, the $\tilde{\mathrm{LS}}$-rank of $P_{n,\rho}$ is n.*

Proof. Let $P := P_{n,\rho}$ and $\ell := n - \lceil \rho \rceil$ (so $P_I = P_{n,\ell}$). We prove our claim by showing that $\max \left\{ \bar{e}^{\top} x : x \in \tilde{\mathrm{LS}}^{n-1}(P) \right\} > \ell$.

First, let $S = [n-1]$, and define $\epsilon := \min \left\{ \lceil \rho \rceil - \rho, \frac{\ell}{n-1} \right\}$. Also, given $T \subseteq S$, let χ_T denote the incidence vector of T in $\{0,1\}^{n-1}$. Now consider the point

$$\bar{x} := \left(\sum_{T \subseteq S, |T| = \ell} \frac{n-\ell}{\binom{n-1}{\ell}(n - \epsilon(n-1))} \begin{pmatrix} \chi_T \\ \epsilon \end{pmatrix} \right) + \left(\sum_{T \subseteq S, |T| = \ell - 1} \frac{\ell - \epsilon(n-1)}{\binom{n-1}{\ell-1}(n - \epsilon(n-1))} \begin{pmatrix} \chi_T \\ 1 \end{pmatrix} \right).$$

First, observe that $\bar{x}$ is a linear combination of the points whose entries in S are integral. Also, $\begin{pmatrix} \chi_T \\ \epsilon \end{pmatrix} \in P$ for all T of size ℓ (by the choice of ϵ), and $\begin{pmatrix} \chi_T \\ 1 \end{pmatrix} \in P$ for all T of size $\ell - 1$ as well.

Furthermore, since $\epsilon(n-1) \leq \ell$, the weights on these points are nonnegative, and do sum up to 1. Thus, $\bar{x}$ is indeed a convex combination of these points. By the symmetry of P and the definition of $\tilde{\mathrm{LS}}$, we can express $\bar{x}$ as a similar convex combination of points in P for all other sets S of size $n-1$. Thus, this shows that $\bar{x} \in \tilde{\mathrm{LS}}^{n-1}(P)$.

On the other hand, it is easy to check that $\bar{x} = \frac{\ell(1-\epsilon)+\epsilon}{n(1-\epsilon)+\epsilon} \bar{e}$, and thus $\bar{e}^{\top} \bar{x} > \ell$ and $\bar{x} \notin P_I$. Hence, we deduce that P has $\tilde{\mathrm{LS}}$-rank n. $\square$

Thus, we see that when ρ is close to $n-1$, the positive semidefiniteness constraint imposed by SA_+ is in fact helpful in generating the desired facet of the integer hull that can be elusive to a weaker polyhedral operator until the n^{th} iteration. In contrast, when $\rho \in (0,1)$, the SA_+- and $\tilde{\mathrm{LS}}$-rank of $P_{n,\rho}$ are both n. In fact, we will show in Section 4 that when $\rho \in (0,1)$, $\mathrm{SA}_+^k(P_{n,\rho})$ and $\tilde{\mathrm{LS}}^k(P_{n,\rho})$ have exactly the same integrality gap (with respect to the direction of the all-ones vector) for every $k \geq 1$.

We next give a lower bound on the SA_+-rank of $P_{n,\rho}$ for some cases where $\rho > 1$, which will be useful when we later establish a BZ'_+-rank lower bound for some of these polytopes. We first need the following result. Suppose $P \subseteq [0,1]^n$. Given $x \in P$, let

$$S(x) := \{ i \in [n] : 0 < x_i < 1 \}.$$

Also, given $x \in [0,1]^n$ and two disjoint sets of indices $I, J \subseteq [n]$, we define the vector $x_J^I \in [0,1]^n$ where

$$x_J^I[i] := \begin{cases} 1 & \text{if } i \in I; \\ 0 & \text{if } i \in J; \\ x[i] & \text{otherwise.} \end{cases}$$

In other words, x_J^I is the vector obtained from x by setting all entries indexed by elements in I to 1, and all entries indexed by elements in J to 0. Then we have the following useful property that is inherited by a wide class of lift-and-project operators.

Lemma 6 (Theorem 15 in [AT16]). *Let $P \subseteq [0,1]^n$ and $x \in P$. If $x_J^I \in P$ for all $I, J \subseteq S(x)$ such that $|I| + |J| \leq k$, then $x \in \mathrm{SA}_+^k(P)$.*

Using Lemma 6, we have the following for the SA_+-rank of $P_{n,\rho}$:

Proposition 7. *For every $n \geq 2$, if $\rho \in (0,n)$ is not an integer and $k < \frac{n(\lceil \rho \rceil - \rho)}{\lceil \rho \rceil}$, then the SA_+-rank of $P_{n,\rho}$ is at least $k+1$.*

Proof. First, observe that

$$k < \frac{n(\lceil \rho \rceil - \rho)}{\lceil \rho \rceil} \iff (n-k)\left(\frac{n - \lceil \rho \rceil}{n} \right) + k < n - \rho.$$

Thus, there exists $\ell \in \mathbb{R}$ such that $(n-k)\ell + k < n - \rho$ and $\ell > \frac{n-\lceil \rho \rceil}{n}$. Consider the point $\bar{x} := \ell \bar{e}$. Since $\ell > \frac{n-\lceil \rho \rceil}{n}$, $\bar{x} \notin (P_{n,\rho})_I$. However, for every pair of disjoint sets of indices $I, J \subseteq [n]$ where $|I| + |J| \leq k$, we have

$$\sum_{i=1}^{n} \bar{x}_J^I[i] \leq (n-k)\ell + k < n - \rho,$$

by the choice of ℓ. Thus, $\bar{x}_J^I \in P_{n,\rho}$ for all such choices of I, J. (Note that the first inequality above follows from the fact that $\sum_{i=1}^{n} \bar{x}_J^I[i]$ is maximized by choosing I, J where $|I| = k$ and $J = \emptyset$.) Thus, it follows from Lemma 6 that $\bar{x} \in \mathrm{SA}_+^k(P_{n,\rho})$. This proves that $\mathrm{SA}_+^k(P_{n,\rho}) \neq (P_{n,\rho})_I$, and hence the SA_+-rank of $P_{n,\rho}$ is at least $k+1$. ☐

Using Proposition 7, we obtain a lower-bound result on the BZ'_+-rank of $P_{n,\rho}$, establishing what we believe to be the first example in which BZ'_+ (and, as a result, BZ_+) requires more than a constant number of iterations to return the integer hull of a set.

Theorem 8. *Suppose an integer $n \geq 5$ is not a perfect square. Then there exists $\rho \in (\lfloor \sqrt{n} \rfloor, \lceil \sqrt{n} \rceil)$ such that the BZ'_+-rank of $P_{n,\rho}$ is at least $\left\lfloor \frac{\sqrt{n}+1}{2} \right\rfloor$.*

Proof. Let $P := P_{n,\rho}$. First, choose $\epsilon \in (0,1)$ small enough such that

$$\sqrt{n} - 1 < \frac{n(1-\epsilon)}{\lceil \sqrt{n} \rceil},$$

and let $\rho := \lfloor \sqrt{n} \rfloor + \epsilon$. Next, let $k := \left\lfloor \frac{\sqrt{n}-1}{2} \right\rfloor$. Notice that for all $n \geq 5$, $k+1 < \rho < n-(k+1)$, and so BZ'^k_+ does not generate any k-small obstructions for P. Thus, we obtain that $\mathrm{SA}_+^{2k}(P) \subseteq \mathrm{BZ}'^k_+(P)$ by Proposition 2. Also, from Proposition 7, since $2k \leq \sqrt{n} - 1$, $\mathrm{SA}_+^{2k}(P) \neq P_I$. Thus, the BZ'_+-rank of P is at least $k + 1 = \left\lfloor \frac{\sqrt{n}+1}{2} \right\rfloor$. ☐

We note that the BZ_+-rank of $P_{n,\rho}$ is 1 for every $\rho \in (0,1)$. This is because the set $[n]$ is a k-small obstruction for every $k \geq 1$, and so $\sum_{i=1}^{n} x_i \leq n-1$ is valid for $\mathcal{O}_k(P_{n,\rho})$, and the refinement step in BZ_+ already suffices in generating the integer hull of $P_{n,\rho}$. More generally, when $k+1 \geq \rho$, every subset of set of $[n]$ of size $n-k$ does qualify as a k-small obstruction, and it can be shown that $\mathrm{BZ}_+^k(P_{n,\rho}) = (P_{n,\rho})_I$. On the other hand, since BZ_+ (and the refined version BZ'_+) dominates SA_+, Proposition 4 implies that the BZ_+-rank of $P_{n,\rho}$ is at most $n - \lceil \rho \rceil + 1$. This implies that, in contrast with other operators (including SA_+ and, as we will see, Las), the BZ_+-rank of $P_{n,\rho}$ is low both when ρ is close to 0 or n.

We next turn to the Las-rank of $P_{n,\rho}$. Interestingly, Cheung showed the following in [Che07]:

Theorem 9. *(i) For every even integer $n \geq 4$, the* Las*-rank of $P_{n,\rho}$ is at most $n-1$ for all $\rho \geq \frac{1}{n}$;*
(ii) For every integer $n \geq 2$, there exists $\rho \in \left(0, \frac{1}{n}\right)$ such that the Las*-rank of $P_{n,\rho}$ is n.*

Thus, while the rank of $P_{n,\rho}$ is invariant under the choice of $\rho \in (0,1)$ with respect to all other lift-and-project operators we have considered so far, it is not the case for Las. Next, we strengthen part (ii) of Cheung's result above, and give a range of ρ where $P_{n,\rho}$ has Las-rank n for every $n \geq 2$.

Theorem 10. *Suppose $n \geq 2$, and*

$$0 < \rho \leq \frac{n^2 - 1}{2n^{n+1} - n^2 - 1}.$$

Then $P_{n,\rho}$ has Las*-rank n.*

Before we prove Theorem 10, we need some notation and lemmas. Define the matrix $Z \in \mathbb{R}^{\mathcal{A}_n^+ \times \mathcal{A}_n^+}$ where

$$Z[S|_1, T|_1] := \begin{cases} 1 & \text{if } S \subseteq T; \\ 0 & \text{otherwise.} \end{cases}$$

Z is the *zeta matrix* of $[n]$. Note that Z is invertible, and it is well known that its inverse is the *Möbius matrix* $M \in \mathbb{R}^{\mathcal{A}_n^+ \times \mathcal{A}_n^+}$ where

$$M[S|_1, T|_1] := \begin{cases} (-1)^{|T \setminus S|} & \text{if } S \subseteq T; \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this paper, we will assume that the rows and columns in Z and M are ordered such that the last row/column corresponds to the set $[n]|_1$. Note that, with such an ordering, the last column of Z is the all-ones vector. The following relation between zeta matrices and moment matrices is due to Laurent (see proof of Lemma 2 in [Lau03]):

Lemma 11. *Suppose $y \in \mathbb{R}^{\mathcal{A}_n^+}$. Define $u \in \mathbb{R}^{\mathcal{A}_n^+}$ where*

$$u[S|_1] := \sum_{T \supseteq S} (-1)^{|T \setminus S|} y[T|_1].$$

Then $\mathcal{M}_n(y) = Z \operatorname{Diag}(u) Z^\top$.

Note that we used $\operatorname{Diag}(u)$ to denote the diagonal matrix U where $U[S|_1, S|_1] := u[S|_1]$ for all $S \subseteq [n]$. Next, the following lemma will be useful for proving Theorem 10, as well as analyzing the cropped hypercube $Q_{n,\rho}$ later on. Note that it uses very similar ideas to that in [KLM15], where they characterized general conditions for when $\mathcal{M}_{n-1}(w)$ is positive semidefinite, although the proof here is simpler as we are specifically focused on the applications to the sets $P_{n,\rho}$ and $Q_{n,\rho}$.

Lemma 12. *Let $\theta \in (0,1)$ be a fixed number. Define $y \in \mathbb{R}^{\mathcal{A}_n^+}$ such that $y[S|_1] := \theta^{|S|}$ for all $S \subseteq [n]$. Then*

(i) $\mathcal{M}_i(y) \succeq 0$ *for all* $i \in [n]$.

(ii) Given any $\rho > 0$,

$$(n-\rho)\mathcal{M}_n(y)[S|_1, T|_1] - \sum_{j=1}^{n} \mathcal{M}_n(y)[(S \cup \{j\})|_1, (T \cup \{j\})|_1] = \mathcal{M}_n(w)[S|_1, T|_1]$$

where $w|S|_1] := ((n - |S|)(1-\theta) - \rho)\,\theta^{|S|}$ *for all* $S \subseteq [n]$. *Moreover,*

$$\mathcal{M}_n(w) = Z \operatorname{Diag}(u) Z^\top,$$

where $u[S|_1] := (n - |S| - \rho)\theta^{|S|}(1-\theta)^{n-|S|}$ *for all* $S \subseteq [n]$.

(iii) If $\rho \in (0,1)$ *and*

$$\rho \leq \frac{(n+1)\theta(1-\theta)^n}{2 - [(n-1)\theta + 2]\,(1-\theta)^n}, \tag{12}$$

then $\mathcal{M}_{n-1}(w) \succ 0$.

Proof. To prove part (i), it suffices to show that $\mathcal{M}_n(y) \succeq 0$, as $\mathcal{M}_i(y)$ is a symmetric minor of $\mathcal{M}_n(y)$ for all $i < n$. By Lemma 11, Since $\mathcal{M}_n(y) = Z \operatorname{Diag}(v) Z^\top$, where

$$v[S|_1] = \sum_{T \supseteq S} (-1)^{|T \setminus S|} y[T|_1] = \sum_{i=0}^{n-|S|} \binom{n-|S|}{i} (-1)^i \theta^{|S|+i} = \theta^{|S|}(1-\theta)^{n-|S|},$$

which is positive for all $S \subseteq [n]$. Thus, it follows that $\mathcal{M}_n(y) \succeq 0$. For part (ii), we see that

$$\begin{aligned} & (n-\rho)\mathcal{M}_n(y)[S|_1, T|_1] - \sum_{i=1}^{n} \mathcal{M}_n(y)[(S \cup \{j\})|_1, (T \cup \{j\})|_1] \\ = \; & (n-\rho)\theta^{|S\cup T|} - \left(|S \cup T|\theta^{|S\cup T|} + (n - |S \cup T|)\theta^{|S\cup T|+1}\right) \\ = \; & \left((n - |S \cup T|)(1-\theta) - \rho\right)\theta^{|S\cup T|} \\ = \; & \mathcal{M}_n(w)[S|_1, T|_1]. \end{aligned}$$

Also, it is not hard to check that $\sum_{T\supseteq S} (-1)^{|T\setminus S|} w[S|_1] = u[S|_1]$ for all $S \subseteq [n]$, and so the last part of the claim follows from Lemma 11.

Finally, for (iii), let $\bar{Z}$ and $\bar{M}$, respectively, denote the symmetric minor of Z and M with the row and column corresponding to $[n]|_1$ removed. We also let $u' \in \mathcal{A}_+^{n-1}$ denote the vector obtained from u by removing the entry corresponding to $[n]|_1$. Then by Lemma 11,

$$\begin{aligned} \mathcal{M}_n(w) = Z \operatorname{Diag}(u) Z^\top &= \begin{pmatrix} \bar{Z} & \bar{e} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \operatorname{Diag}(u') & 0 \\ 0 & -\theta^n\rho \end{pmatrix} \begin{pmatrix} \bar{Z}^\top & 0 \\ \bar{e}^\top & 1 \end{pmatrix} \\ &= \begin{pmatrix} \bar{Z}\operatorname{Diag}(u')\bar{Z}^\top - \theta^n\rho\bar{e}\bar{e}^\top & -\theta^n\rho\bar{e} \\ -\theta^n\rho\bar{e}^\top & -\theta^n\rho \end{pmatrix}. \end{aligned}$$

Since $\mathcal{M}_{n-1}(w)$ is the symmetric minor of $\mathcal{M}_n(w)$ with the last row and column removed, we obtain that $\mathcal{M}_{n-1}(w) = \bar{Z}\operatorname{Diag}(u')\bar{Z}^\top - \theta^n\rho\bar{e}\bar{e}^\top$. Notice that $u[S|_1] = (n-|S|-\rho)\theta^{|S|}(1-\theta)^{n-|S|} > 0$ for all $S \subset [n]$, and that $\bar{M}$ is nonsingular ($\bar{M}$ is the inverse of $\bar{Z}$). Hence, $[\operatorname{Diag}(u')]^{-1/2}\bar{M}$ is nonsingular and $[\operatorname{Diag}(u')]^{-1/2}\bar{M} \cdot \bar{M}^\top [\operatorname{Diag}(u')]^{-1/2}$ is an automorphism of the underlying cone of positive semidefinite matrices. Therefore, $\mathcal{M}_{n-1}(w) \succ 0$ if and only if

$$Y := \left[\operatorname{Diag}(u')\right]^{-1/2} \bar{M}\mathcal{M}_{n-1}(w)\bar{M}^\top \left[\operatorname{Diag}(u')\right]^{-1/2}$$

is positive definite. Now observe that $Y = I - \rho\theta^n\xi\xi^\top$, where $\xi := [\operatorname{Diag}(u')]^{-1/2}\bar{M}\bar{e}$. Hence,

$$\mathcal{M}_{n-1}(w) \succ 0 \iff \rho\theta^n\xi^\top\xi < 1. \tag{13}$$

Next, using the fact that $(\bar{M}\bar{e})[S|_1] = (-1)^{n-|S|-1}$ for all $S \subset [n]$, we analyze $\rho\theta^n\xi^\top\xi$ which is equal to:

$$\begin{aligned} \rho\theta^n\left(\sum_{S\subset[n]} \frac{1}{u[S|_1}\right) &= \rho\theta^n\left(\sum_{S\subset[n]} \frac{1}{(n-|S|-\rho)\theta^{|S|}(1-\theta)^{n-|S|}}\right) \\ &= \frac{\rho\theta^n}{(1-\theta)^n}\left(\sum_{i=0}^{n-1} \frac{1}{n-i-\rho}\binom{n}{i}\left(\frac{1-\theta}{\theta}\right)^i\right) \\ &< \frac{\rho\theta^n}{(1-\theta)^n}\left(\sum_{i=0}^{n-1} \frac{2}{(n+1)(1-\rho)}\binom{n+1}{i}\left(\frac{1-\theta}{\theta}\right)^i\right) \\ &= \frac{2\rho\theta^n}{(1-\rho)(n+1)(1-\theta)^n}\left(\left(\frac{1}{\theta}\right)^{n+1} - (n+1)\left(\frac{1-\theta}{\theta}\right)^n - \left(\frac{1-\theta}{\theta}\right)^{n+1}\right) \\ &= \frac{\rho}{1-\rho}\left(\frac{2\left[1-(n\theta+1)(1-\theta)^n\right]}{(n+1)\theta(1-\theta)^n}\right). \end{aligned}$$

Thus, if $\rho \leq \frac{(n+1)\theta(1-\theta)^n}{2-[(n-1)\theta+2](1-\theta)^n}$, then $\mathcal{M}_{n-1}(w)$ is positive definite. $\square$

We are now ready to prove Theorem 10.

Proof of Theorem 10. It is obvious that $(P_{n,\rho})_I = P_{n,1}$ for all $\rho \in (0,1)$. Now suppose we are given integer $n \geq 2$ and $0 < \rho \leq \frac{n^2-1}{2n^{n+1}-n^2-1}$. We prove our claim by showing that there exists $\theta > \frac{n-1}{n}$ where $\theta\bar{e} \in \mathrm{Las}^{n-1}(P_{n,\rho})$.

Define $y \in \mathbb{R}^{\mathcal{A}_n^+}$ where $y[S|_1] := \theta^{|S|}$ for all $S \subseteq [n]$, then Lemma 12 implies $Y := \mathcal{M}_n(y) \succeq 0$. It also implies that $Y^1 = \mathcal{M}_{n-1}(w)$ where $w|S|_1] = ((n-|S|)(1-\theta)-\rho)\theta^k$. To prove our claim, it suffices to show that there exists $\theta > \frac{n-1}{n}$ such that $\mathcal{M}_{n-1}(w) \succeq 0$. Since our upper bound on ρ is continuous in θ in a neighbourhood of $\theta = \frac{n-1}{n}$, and the cone of positive definite matrices is the interior of the cone of positive semidefinite matrices, by (13), it suffices to show that $\mathcal{M}_{n-1}(w) \succ 0$ when $\theta = \frac{n-1}{n}$. Then by letting $\theta = \frac{n-1}{n}$ in (12) and simplifying, we obtain that $\rho \leq \frac{n^2-1}{2n^{n+1}-n^2-1}$ guarantees $\mathcal{M}_{n-1}(w) \succ 0$, and the claim follows. $\square$

Let $p(n)$ denote the largest $\rho > 0$ where $\mathcal{M}_{n-1}(y) \in \widehat{\mathrm{Las}}^{n-1}(P_{n,\rho})$ for some $\theta > \frac{n-1}{n}$ (where y is defined in the proof of Theorem 10). Figure 3 shows the value of $\log_n(p(n))$ for some small values of n, as well as the lower bound on $p(n)$ given by Theorem 10.

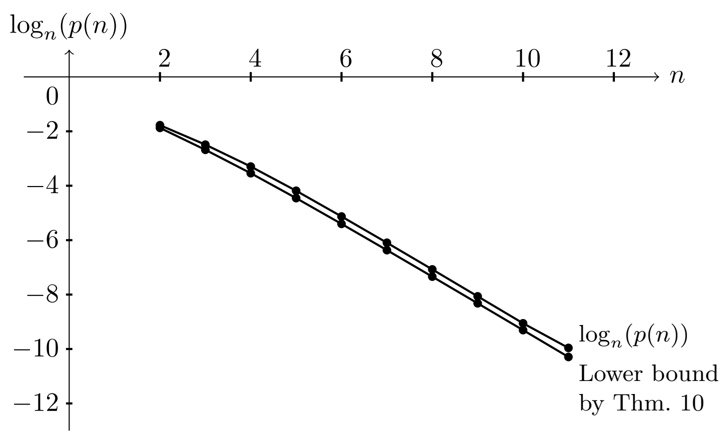

FIGURE 3. Computational results and lower bounds for $p(n)$.

3.2. **The cropped hypercube $Q_{n,\rho}$.** Next, we turn our attention to the cropped hypercube

$$Q_{n,\rho} := \left\{ x \in [0,1]^n : \sum_{i \in S}(1-x_i) + \sum_{i \notin S} x_i \geq \rho, \ \forall S \subseteq [n] \right\}.$$

Observe that, for every $S \subseteq [n]$, its incidence vector violates the inequality corresponding to S in the description of $Q_{n,\rho}$. Thus, we see that $(Q_{n,\rho})_I = \emptyset$. Independently, Cook and Dash [CD01] and Goemans and the second author [GT01] showed that $Q_{n,1/2}$ has LS_+-rank n. Subsequently, the authors showed in [AT16] that the SA_+-rank of $Q_{n,1/2}$ is also n. In fact, the results therein readily imply that $\mathrm{SA}_+^k(Q_{n,\rho}) = Q_{n,\rho-k/2}$ for all $\rho \in (0,1/2]$ and $k \in [n]$. Thus, it follows that $Q_{n,\rho}$ has SA_+-rank n for all $\rho \in (0,1/2]$.

More generally, Pokutta and Schulz [PS10] showed that $Q_{n,1/2}$ has rank $\Omega(n/\log(n))$ with respect to any admissible cutting-plane procedure, which is a very broad framework that encompasses many well-known methods of generating cutting planes, such as Gomory-Chvátal cuts, split cuts, as well as all lift-and-project operators mentioned herein except for the Bienstock–Zuckerberg variants. Subsequently, instead of applying an admissible cutting-plane procedure Γ to a set P and obtain a tightened relaxation $\Gamma(P)$, Dey and Pokutta [DP14] studied the verification closure of this procedure — a mechanism that yields a yet stronger (but generally intractable) relaxation $\delta\Gamma(P)$. Their results imply that $Q_{n,1/2}$ has rank $\Omega(n)$ with respect to $\delta\,\mathrm{LS}_+$ (the stronger, verification version of LS_+). More recently, Bodur, Dash, and G´unl´uk [BDG17] studied extended LP formulations of a given set, and showed that there exists a set $\hat{Q} \subseteq \mathbb{R}^{2n-1}$ where the projection of $\hat{Q}$ onto the first n variables is $Q_{n,1/2}$, and that $\mathrm{LS}_0(\hat{Q}) = \emptyset$. That is, while LS_0 requires n iterations to return the (empty) integer hull when directly applied to $Q_{n,1/2}$, applying it to a suitably constructed extended formulation with $(n-1)$ extra variables allows LS_0 to reach the integer hull in just one iteration.

We now turn to the Las-rank of the cropped hypercubes. First, $Q_{n,1/2}$ has been shown to have Las-rank 1 for $n = 2$ [Lau03], and 2 for $n = 4$ [Che07]. While Las depends on the algebraic description of the initial relaxation, the following observation significantly simplifies the analysis of the Las-rank of $Q_{n,\rho}$.

Proposition 13. *Suppose n, k are fixed positive integers and $\rho \in (0, 1)$. Define the vector $w \in \mathbb{R}^{\mathcal{A}_n^+}$ where*

$$w[S|_1] := (n - |S| - 2\rho)2^{-|S|-1}, \ \forall S \subseteq [n].$$

Then $\mathrm{Las}^k(Q_{n,\rho}) \neq \emptyset$ if and only if $\mathcal{M}_k(w) \succeq 0$.

Proof. Suppose $\mathrm{Las}^k(Q_{n,\rho}) \neq \emptyset$, and let $Y \in \widehat{\mathrm{Las}}^k(Q_{n,\rho})$. Notice that every automorphism for the unit hypercube is also an automorphism for $Q_{n,\rho}$. If we take these $2^n n!$ automorphisms and apply them onto Y as outlined in the proof of Proposition 1, we obtain $2^n n!$ matrices in $\widehat{\mathrm{Las}}^k(Q_{n,\rho})$. Let $\bar{Y}$ be the average of these matrices. Then by the symmetry of $Q_{n,\rho}$, we know that $\bar{Y} = \mathcal{M}_k(y)$, where $y[S|_1] = 2^{-|S|}, \ \forall S \subseteq [n]$.

By the convexity of $\widehat{\mathrm{Las}}^k(Q_{n,\rho})$, $\bar{Y} \in \widehat{\mathrm{Las}}^k(Q_{n,\rho})$, and thus satisfies (Las 2) for all of the 2^n equalities defining $Q_{n,\rho}$. In fact, due to the entries of $\bar{Y}$, the matrix $\bar{Y}^j$ is the same for all 2^n inequalities describing $Q_{n,\rho}$. Thus, using the inequality $\sum_{i=1}^n x_i \leq n - \rho$ and applying Lemma 12 with $\theta = \frac{1}{2}$, we obtain that

$$\bar{Y}^j[S|_1, T|_1] = (n - |S \cup T| - 2\rho)2^{-|S\cup T|-1} = \mathcal{M}_k(w)[S|_1, T|_1]$$

for all $S, T \subseteq [n], |S|, |T| \leq k$. Hence, we deduce that $\mathrm{Las}^k(Q_{n,\rho}) \neq \emptyset \Rightarrow \mathcal{M}_k(w) \succeq 0$.

The converse can be proven by tracing the above argument backwards. First, it follows from Lemma 12 that $\bar{Y} \succeq 0$. Then, again, the matrix $\bar{Y}^j$ is exactly $\mathcal{M}_k(w)$ for all 2^n inequalities describing $Q_{n,\rho}$. Since $\mathcal{M}_k(w) \succeq 0$ by assumption, $\bar{Y} \in \widehat{\mathrm{Las}}^k(Q_{n,\rho})$. Thus, we obtain that $\frac{1}{2}\bar{e} \in \mathrm{Las}^k(Q_{n,\rho})$, and so $\mathrm{Las}^k(Q_{n,\rho}) \neq \emptyset$. □

Thus, computing the Las-rank of $Q_{n,\rho}$ reduces to finding the largest k where the matrix $\mathcal{M}_k(w)$ defined in the statement of Proposition 13 is positive semidefinite (which would then imply that the Las-rank of $Q_{n,\rho}$ is $k + 1$). Using that, we are able to show the following:

Theorem 14. *For every $n \geq 2$, let $q(n)$ be the largest ρ where $Q_{n,\rho}$ has* Las*-rank n. Then*

$$\frac{n+1}{2^{n+2} - n - 3} \leq q(n) \leq \frac{n}{2^{n+1} - 2}.$$

Proof. We first prove the lower bound. If we let $\theta = \frac{1}{2}$ in (12), we obtain that $\rho \leq \frac{n+1}{2^{n+2}-n-3}$ implies $\mathcal{M}_{n-1}(w) \succeq 0$ where $w[S|_1] = (n - |S| - 2\rho)2^{-|S|-1}$, $\forall S \subseteq [n]$. Thus, the claim follows from Proposition 13.

As for the upper bound, we show that if $\rho > \frac{n}{2^{n+1}-2}$, then $Q_{n,\rho}$ has Las-rank at most $n-1$. Define $x \in \mathbb{R}^{\mathcal{A}_n^+}$ where $x[S|_1] := (-2)^{|S|}$ for all $S \subseteq [n]$. Also, let x' denote the vector in $\mathbb{R}^{\mathcal{A}_{n-1}^+}$ obtained from x by removing the entry corresponding to $[n]|_1$. By Proposition 13, if we let $w[S|_1] = (n - |S| - 2\rho)2^{-|S|-1}$ for all $S \subseteq [n]$ and show that $x'^\top \mathcal{M}_{n-1}(w) x' < 0$ whenever $\rho > \frac{n}{2^{n+1}-2}$, then $\mathcal{M}_{n-1}(w) \not\succeq 0$, and our claim follows.

Recall that $\mathcal{M}_n(w) = Z \operatorname{Diag}(u) Z^\top$ where $u[S|_1] = (n - |S| - \rho)2^{-n}$ for all $S \subseteq [n]$. Also, note that $(Z^\top x)[S|_1] = (-1)^{|S|}$ for all $S \subseteq [n]$. Thus,

$$
\begin{aligned}
x^\top \mathcal{M}_n(w) x &= x^\top Z \operatorname{Diag}(u) Z^\top x \\
&= \sum_{S \subseteq [n]} \left((Z^\top x)[S|_1] \right)^2 u[S|_1] \\
&= \sum_{i=0}^{n} \binom{n}{i} \left((-1)^i \right)^2 (n - i - \rho) 2^{-n} \\
&= \left(n2^n - n2^{n-1} - \rho 2^n \right) 2^{-n} \\
&= \frac{n}{2} - \rho.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
x^\top \mathcal{M}_n(w) x &= \begin{pmatrix} x'^\top & (-2)^n \end{pmatrix} \begin{pmatrix} \mathcal{M}_{n-1}(w) & -2^{-n}\rho\bar{e} \\ -2^{-n}\rho\bar{e}^\top & -2^{-n}\rho \end{pmatrix} \begin{pmatrix} x' \\ (-2)^n \end{pmatrix} \\
&= \begin{pmatrix} x'^\top \mathcal{M}_{n-1}(w) + (-1)^{n+1}\rho\bar{e}^\top & -2^{-n}\rho x'^\top \bar{e} + (-1)^{n+1}\rho \end{pmatrix} \begin{pmatrix} x' \\ (-2)^n \end{pmatrix} \\
&= x'^\top \mathcal{M}_{n-1}(w) x' + (-1)^{n+1}\rho\bar{e}^\top x' + (-1)^{n+1}\rho x'^\top \bar{e} + (-1)^{n+1}(-2)^n \rho \\
&= x'^\top \mathcal{M}_{n-1}(w) x' + (2^n - 2)\rho.
\end{aligned}
$$

(It is helpful to observe that $\bar{e}^\top x' = (-1)^n - (-2)^n$.) Hence, we combine the above and obtain that

$$
x'^\top \mathcal{M}_{n-1}(w) x' = \frac{n}{2} - (2^n - 1)\rho,
$$

which is negative whenever $\rho > \frac{n}{2^{n+1}-2}$. This finishes the proof. □

Therefore, akin to what Cheung showed for $P_{n,\rho}$, there does not exist a fixed ρ where $Q_{n,\rho}$ has Las-rank n for all n. Also, as with $P_{n,\rho}$, the Las-rank of $Q_{n,\rho}$ varies under the choice of ρ. For instance, Figure 4 illustrates the Las-rank for $Q_{n,\ell/1000}$ for $\ell \in [500]$ and several values of n. The pattern is similar for all other values of n we were able to test — the Las-rank is around $\frac{n}{2}$ when $\rho = \frac{1}{2}$, and slowly rises to n as ρ approaches 0. Recently, related to the Figure 4, Kurpisz, Leppänen and Mastrolilli [KLM16] proved that the Lasserre rank of $Q_{n,1/2}$ is between $\Omega(\sqrt{n})$ and $n - \Omega(n^{1/3})$.

Also, recall that we let $q(n)$ be the largest ρ where $Q_{n,\rho}$ has Las-rank n. It follows from Theorem 14 that $\frac{2^{n+1}}{nq(n)}$ is roughly bounded between $\frac{1}{2}$ and 1. Note that since Las imposes as many positive semidefiniteness constraints as there are defining inequalities for the given relaxation (which there are exponentially many for $Q_{n,\rho}$), the relaxation $\text{Las}^k(Q_{n,\rho})$ is not obviously tractable, even when k is a constant. Now, computing $q(n)$ requires verifying whether

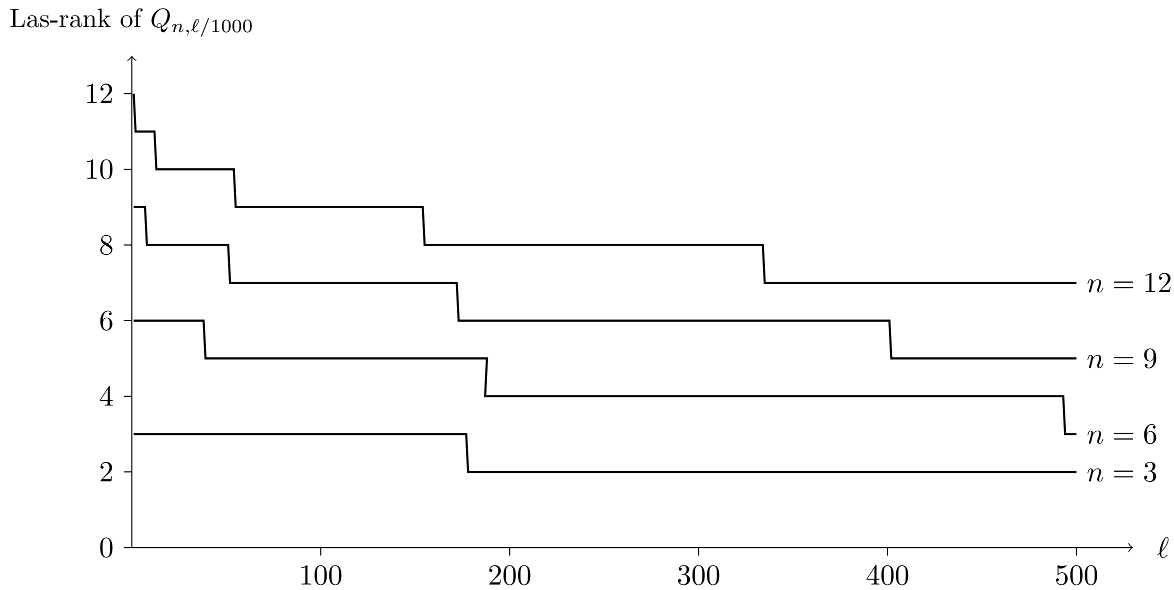

FIGURE 4. The Las-rank of $Q_{n,\rho}$ for varying values of $\rho := \ell/1000$, for $n \in \{3, 6, 9, 12\}$.

$\text{Las}^{n-1}(Q_{n,\rho})$ is empty, which by definition of Las is the projection of $\widehat{\text{Las}}^{n-1}(Q_{n,\rho})$, a set of matrices of order $\Omega(2^n) \times \Omega(2^n)$ with $\Omega(2^n)$ positive semidefiniteness constraints. Instead of solving the feasibility problem of such a large number of variables and constraints, Proposition 13 uses the symmetries of $Q_{n,\rho}$ (as well as the fact that Las preserves symmetries and commutes with all automorphisms of the unit hypercube, as shown in Proposition 1) to reduce this task to checking the positive semidefiniteness of $\mathcal{M}_{n-1}(w)$, a $(2^n - 1) \times (2^n - 1)$ matrix with known entries. Furthermore, notice that if $\mathcal{M}_{n-1}(w)$ had an eigenvector x with negative eigenvalue, we could assume that $x[S|_1] = x[T|_1]$ whenever $|S| = |T|$, due to the symmetries of the entries in $\mathcal{M}_{n-1}(w)$. Hence, if we define the n-by-n matrix W whose rows and columns are indexed by $\{0, 1, \ldots, n-1\}$ such that

$$\begin{aligned} W[i,j] &:= \sum_{S,T \subseteq [n], |S|=i, |T|=j} \mathcal{M}_{n-1}(w)[S|_1, T|_1] \\ &= 2^{-i-j-1} n \binom{n-1}{i} \binom{n-1}{j} \left(\sum_{k=0}^{n-1} \frac{\binom{i}{k}\binom{j}{k}}{\binom{n-1}{k}} \right) - \rho 2^{-i-j} \binom{n}{i} \binom{n}{j} \left(\sum_{k=0}^{n} \frac{\binom{i}{k}\binom{j}{k}}{\binom{n}{k}} \right), \end{aligned}$$

then it follows that $\mathcal{M}_{n-1}(w) \succeq 0$ if and only if $W \succeq 0$. This reduction allows us to verify if $Q_{n,\rho}$ has Las-rank n by simply checking if the n-by-n matrix W is positive semidefinite. Using the reduction above, we computed $\frac{2^{n+1}}{nq(n)}$ to within two decimal places for $n \in \{2, 3, \ldots, 16\}$, as illustrated in Figure 5.

As for the BZ'_+-rank of $Q_{n,\rho}$, it was shown in [BZ04] that $Q_{n,1/2}$ has BZ-rank 2, where BZ is a polyhedral operator dominated by BZ_+ and BZ'_+. Thus, it follows that the BZ'_+-rank of $Q_{n,1/2}$ is at most 2. However, we remark that, as with the Lasserre operator, the Bienstock–Zuckerberg operators also require an explicitly given system of inequalities for the input set. In particular, the run-time of these operators depends on the size of the system (which, again, is exponential in n in the case of $Q_{n,\rho}$). Thus, $\text{BZ}^k(Q_{n,\rho})$ is not obviously tractable, even for

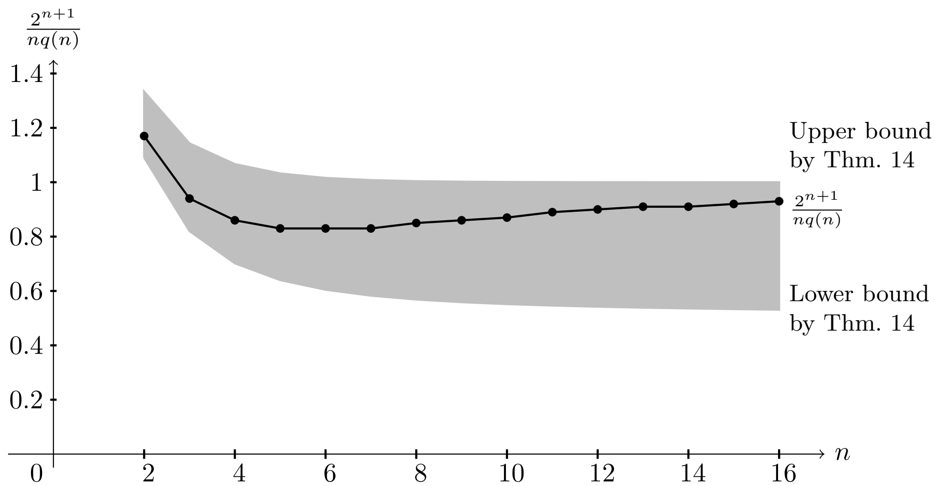

FIGURE 5. Computational results and possible ranges for $q(n) := \min \left\{\rho : \mathrm{Las}^{n-1}\left(Q_{n,\rho}\right) \neq \emptyset\right\}$.

$k = O(1)$. On the other hand, operators such as $\mathrm{SA}_+, \mathrm{SA}$ and BCC are able to produce tightened relaxations that are tractable as long as we have an efficient separation oracle of the input set (which does exist for the cropped hypercube — note that $x \in Q_{n,\rho}$ if and only if $x \in [0,1]^n$ and $\sum_{i=1}^{n} |x_i - \frac{1}{2}| \leq \frac{n}{2} - \rho$).

4. Integrality gaps of lift-and-project relaxations

We conclude this paper by noting some interesting tendencies of the integrality gaps of some lift-and-project relaxations. First, given a compact, convex set $P \subseteq [0,1]^n$ where $P_I \neq \emptyset$ and vector $c \in \mathbb{R}^n$, the *integrality gap* of P with respect to c is defined to be

$$\gamma_c(P) := \frac{\max\left\{c^\top x : x \in P\right\}}{\max\left\{c^\top x : x \in P_I\right\}}.$$

The integrality gap gives a measure of how "tight" the relaxation P is in the objective function direction of c. Here, we show that the integrality gap of $P_{n,\rho}$ with respect to the all-ones is invariant under k iterations of several different operators.

Theorem 15. *For every integer $n \geq 2$, for every $\rho \in (0,1)$ and for every operator $\Gamma \in \left\{\tilde{\mathrm{LS}}, \mathrm{LS}_+, \mathrm{SA}, \mathrm{SA}_+\right\}$, we have*

$$\gamma_{\bar{e}}\left(\Gamma^k\left(P_{n,\rho}\right)\right) = 1 + \frac{(n-k)(1-\rho)}{(n-1)(n-k+k\rho)},$$

for every $k \in \{0,1,2,\ldots,n\}$.

Proof. We prove our claim by showing that

$$\max\left\{\theta : \theta\bar{e} \in \tilde{\mathrm{LS}}^k\left(P_{n,\rho}\right)\right\} \geq \frac{n-k+(k-1)\rho}{n-k+k\rho} \geq \max\left\{\theta : \theta\bar{e} \in \mathrm{SA}_+^k\left(P_{n,\rho}\right)\right\}. \tag{14}$$

Then the result follows from the dominance relationships between the operators. First, the claim is obvious when $k = 0$ or when $k = n$, and thus from here on we assume that $k \in [n-1]$. Let $P := P_{n,\rho}$. We first prove the first inequality in (14).

Given $\theta\bar{e} \in \tilde{\mathrm{LS}}^k(P)$, we know that there exist coefficients a_T and vectors $v_T \in [0,1]^{n-k}$ for each $T \subseteq [k]$ where

$$\theta\bar{e} = \sum_{T\subseteq[k]} a_T \begin{pmatrix} \chi_T \\ v_T \end{pmatrix}.$$

(Here, χ_T is the incidence vector of T in $\{0,1\}^k$.) Note that the operator $\tilde{\mathrm{LS}}$ requires that the a_T's be nonnegative and sum up to 1. Also, note that $\begin{pmatrix} \chi_T \\ v_T \end{pmatrix}$ is in P for all $v_T \in [0,1]^{n-k}$ whenever $|T| < k$. For $T = [k]$, the constraint $\bar{e}^\top x \leq n - \rho$ implies that $\bar{e}^\top v_T \leq n - k - \rho$.

Due to the symmetry of P, given one convex combination of $\theta\bar{e}$, we could obtain many other convex combinations by applying any permutation on $[n]$ that fixes $[k]$. If we take the average of all these combinations, we would obtain a "symmetric" one where $a_T = a_{T'}$ and $v_T = v_{T'}$ whenever $|T| = |T'|$, and that v_T's are all multiples of the all-ones vector. Take such a combination of $\theta\bar{e}$, and define

$$a_i := \sum_{T\subseteq[k], |T|=i} a_T, \quad v_i := \frac{1}{\binom{k}{i}} \sum_{T\subseteq[k],|T|=i} v_T$$

for every $i = 0, 1, \ldots, k$. Then observe that

$$\theta\bar{e} = \sum_{i=0}^{k} a_i \begin{pmatrix} \frac{i}{k}\bar{e} \\ v_i\bar{e} \end{pmatrix}. \tag{15}$$

Moreover, note that the a_i's are nonnegative and sum to 1, $0 \leq v_i \leq 1$ for all $i < k$, and $0 \leq v_k \leq \frac{n-k-\rho}{n-k}$. Thus, (15) is equivalent to saying that the point $(\theta, \theta) \in \mathbb{R}^2$ is a convex combination of the points in the sets $\left\{(\frac{i}{k}, v_i) : i \in \{0,1,\ldots,k-1\}, 0 \leq v_i \leq 1\right\}$ and $\left\{(1, v_i) : 0 \leq v_i \leq \frac{n-k-\rho}{n-k}\right\}$. It is easy to see that the convex hull of these points in $\mathbb{R}^2$ form the polytope illustrated in Figure 6.

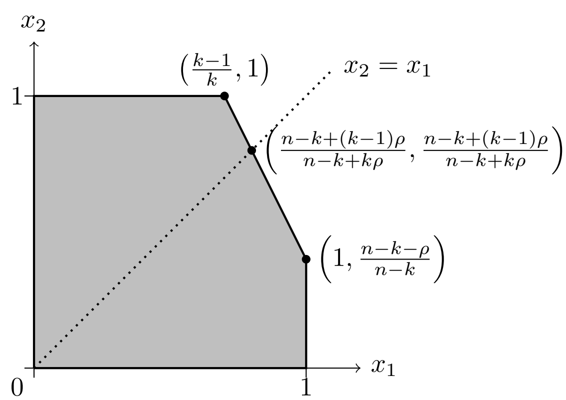

FIGURE 6. Reduction of finding $\max\left\{\theta : \theta\bar{e} \in \tilde{\mathrm{LS}}^k\left(P_{n,\rho}\right)\right\}$ to two dimensions.

Then it is easy to see that the largest θ where (θ, θ) is contained in the convex hull is obtained by the convex combination

$$\frac{n-k}{n-k+k\rho}\begin{pmatrix} 1 \\ \frac{n-k-\rho}{n-k} \end{pmatrix} + \frac{k\rho}{n-k+k\rho}\begin{pmatrix} \frac{k-1}{k} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{n-k+(k-1)\rho}{n-k+k\rho} \\ \frac{n-k+(k-1)\rho}{n-k+k\rho} \end{pmatrix}.$$

This establishes the upper bound on θ.

Next, we turn to show the second inequality in (14) by proving that $\frac{n-k+(k-1)\rho}{n-k+k\rho}\bar{e} \in \mathrm{SA}_+^k(P)$ for every k. First, define $y \in \mathcal{A}_n^+$ where $y[S|_1] := 1 - \frac{|S|\rho}{n-k+k\rho}$ for every $S \subseteq [n]$. We first show that $\mathcal{M}_n(y) \succeq 0$. By Lemma 11, we know that $\mathcal{M}_n(y) = Z\,\mathrm{Diag}(u)Z^\top$ where u is the vector with entries

$$\begin{aligned} u[S|_1] &= \sum_{T \supseteq S} (-1)^{|T \setminus S|} y[S|_1] = \sum_{j=0}^{n-|S|} \binom{n-|S|}{j} (-1)^j \left(1 - \frac{(|S|+j)\rho}{n-k+k\rho}\right) \\ &= \begin{cases} 0 & \text{if } |S| \leq n-2; \\ \frac{\rho}{n-k+k\rho} & \text{if } |S| = n-1; \\ 1 - \frac{n\rho}{n-k+k\rho} & \text{if } S = [n]. \end{cases} \end{aligned}$$

Note that $1 - \frac{n\rho}{n-k+k\rho} \geq 0 \iff (n-k)(\rho-1) \leq 0$, which does hold as $n \geq k$ and $\rho < 1$. Hence, since $u \geq 0$, we deduce that $\mathcal{M}_n(y) \succeq 0$, and in particular $\mathcal{M}_k(y) \succeq 0$.

Next, define $L \in \mathbb{R}^{\mathcal{A}_k \times \mathcal{A}_k^+}$ where

$$L[S|_1 \cap T|_0, U|_1] := \begin{cases} (-1)^{|S|} & \text{if } S \cup T = U; \\ 0 & \text{otherwise,} \end{cases}$$

and let $Y := L\mathcal{M}_k(y)L^\top$. We claim that $Y \in \widehat{\mathrm{SA}}_+^k(P)$. First, $\mathcal{M}_k(y) \succeq 0$ implies that $Y \succeq 0$. Also, (SA_+ 1) holds as $Y[\mathcal{F},\mathcal{F}] = 1$, and it is not hard to see that $Y \geq 0$, as every entry in Y is either $0, \frac{\rho}{n-k+k\rho}$ or $1 - \frac{i\rho}{n-k+k\rho}$ for some integer $i \in \{0,\ldots,n\}$. Next, we check that $\hat{x}(Ye_\beta) \in K(P)$ for all $\beta \in \mathcal{A}_k$. Given $\beta = S|_1 \cap T|_0, \hat{x}(Ye_\beta)$ is the zero vector whenever $|T| \geq 2$, and is the vector $\frac{\rho}{n\rho+1-\rho}(\bar{e}-e_i)$ whenever $T = \{i\}$ for some $i \in [n]$. In both cases, $\sum_{i=1}^n Y[i|_1, \beta] \leq (n-\rho)Y[\mathcal{F},\beta]$ easily holds.

Finally, suppose $\beta = S|_1$ for some $S \subseteq [n]$ where $|S| \leq k$. Observe that

$$Y[\alpha, S|_1] = \begin{cases} \frac{n-k+(k-|S|)\rho}{n-k+k\rho} & \text{if } \alpha = \mathcal{F} \text{ or } \alpha = i|_1 \text{ where } i \in S; \\ \frac{n-k-(k-|S|-1)\rho}{n-k+k\rho} & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} \sum_{i=1}^n Y[i|_1, S|_1] &= |S|\left(\frac{n-k+(k-|S|)\rho}{n-k+k\rho}\right) + (n-|S|)\left(\frac{n-k+(k-|S|-1)\rho}{n-k+k\rho}\right) \\ &\leq (n-\rho)\left(\frac{n-k+(k-|S|)\rho}{n-k+k\rho}\right) \\ &= (n-\rho)Y[\mathcal{F}, S|_1]. \end{aligned}$$

Thus, $\hat{x}(Ye_\beta) \in K(P)$ in this case as well. Finally, it is not hard to see that the entries of Y satisfy (SA_+ 3), (SA_+ 4) and (SA_+ 5). This completes our proof. $\square$

Figure 7 illustrates the integrality gaps of $\mathrm{SA}_+^k(P_{n,\rho})$ for various values of k and ρ in the case $n = 10$ (the behaviour is similar for other values of n). In general, when ρ is close to 1, the gap decreases at an almost-linear rate towards 1. On the other hand, when ρ is small, the integrality gap of $\mathrm{SA}_+^k(P_{n,\rho})$ stays relatively close to $1 + \frac{1}{n-1}$ as k increases to $n-1$, and then abruptly drops to 1 at the n^{th} iteration, where we obtain the integer hull. Again, it follows from Theorem 15 that these gaps would be identical if we replaced SA_+ by any operator Γ where SA_+ dominates Γ and Γ dominates $\tilde{\mathrm{LS}}$.

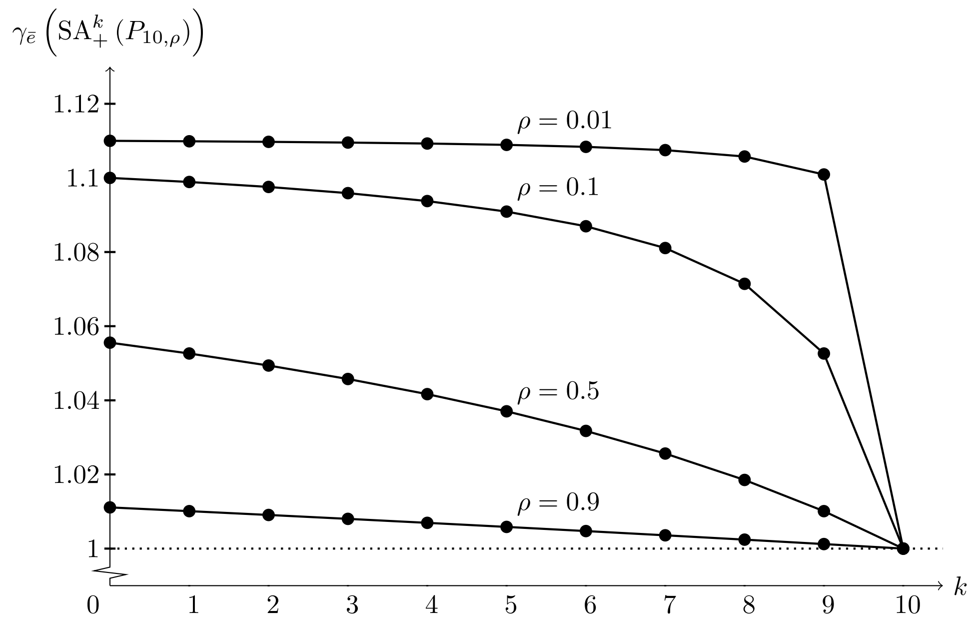

FIGURE 7. Integrality gaps of $\mathrm{SA}_+^k\left(P_{10,\rho}\right)$ for various k and ρ.

We also note that Theorem 15 implies Proposition 3. Moreover, the techniques used for proving the first inequality in (14) can be extended to compute $\max\left\{\theta : \theta\bar{e} \in \tilde{\mathrm{LS}}^k\left(P_{n,\rho}\right)\right\}$ for any non-integer $\rho \in (0,n)$, which would imply Proposition 5.

While the integrality gap for $Q_{n,\rho}$ is undefined (as its integer hull is empty for all $\rho > 0$), we see a similar distinction between its SA_+ and Las relaxations. Note that since all lift-and-project operators we have studied preserve containment, starting with a tighter initial relaxation might offer a lift-and-project operator a head start and yield stronger relaxations in fewer iterations. However, in the case of $Q_{n,\rho}$, different lift-and-project operators utilize this head start in different ways. As mentioned earlier, we know that $\mathrm{SA}_+^k\left(Q_{n,\rho}\right) = Q_{n,\rho-k/2}$ for all $\rho \in (0,1/2]$ and for all $k = [n]$. Thus, given ρ, ρ' where $0 < \rho < \rho' \leq \frac{1}{2}$,

$$\mathrm{SA}_+^k\left(Q_{n,\rho}\right) = Q_{n,(\rho+k/2)} \supset Q_{n,(\rho'+k/2)} = \mathrm{SA}_+^k\left(Q_{n,\rho'}\right),$$

for all $k \in [n-1]$. However, they still converge to the integer hull in the same number of steps. On the other hand, as shown in Theorem 13 and Figure 4, starting with a larger ρ can help Las arrive at the integer hull in fewer iterations, similar to what we saw with $P_{n,\rho}$.

Thus, at least in the case of $P_{n,\rho}$ and $Q_{n,\rho}$ where $\rho \in (0,1)$, all aforementioned operators that are no stronger than SA_+ perform pretty much equally poorly, while deploying Las does achieve some tangible improvements in rank (at least when ρ is not extremely small). Granted, since the number of inequalities imposed by most lift-and-project methods are superpolynomial in n after $\Omega(\log(n))$ rounds, an operator managing to return the integer hull in, say, $\Omega(\sqrt{n})$ iterations is already exerting exponential effort. In that case, claiming that this operator performs better than another that requires (say) $\Omega(n)$ rounds is somewhat a moot point in practice, at the time of this writing.

Of course, there do exist examples where a stronger lift-and-project operator manages to return a tractable relaxation and outperforms exponential effort by a weaker operator: We showed in Propositions 4 and 5 that when $\rho = n - O(1)$, SA_+ would return the integer hull in

$O(1)$ iterations, while $\tilde{\mathrm{LS}}$ requires $\Omega(n)$ rounds. Another such instance is the following: Given a graph $G = (V, E)$, consider its fractional stable set polytope, which is defined as

$$\mathrm{FRAC}(G) = \left\{x \in [0,1]^V : x_i + x_j \leq 1,\ \forall \{i,j\} \in E\right\}.$$

When G is the complete graph on n vertices, it is well known that for hierarchies of polyhedral lift-and-project relaxations (including SA), the integrality gap (with respect to $\bar{e}$) starts at $\frac{n}{2}$, then gradually decreases, and reaches 1 after $\Omega(n)$ iterations. On the other hand, it takes semidefinite operators such as $\mathrm{LS}_+, \mathrm{SA}_+$ and Las exactly one iteration to reach the stable set polytope of K_n, and thus the corresponding integrality gaps for these operators would dive from $\frac{n}{2}$ to 1 in just one iteration.

This raises the natural question of whether, in general, there is some efficient way where we could diagnose a given problem and determine the "best" lift-and-project method for the job. One step in that direction is through studying how various methods perform on different problem classes. Such studies would hopefully provide us better guidance on when it is worthwhile to apply an operator that is more powerful but has a higher per-iteration computational cost.

To take this point further, perhaps one could build a shape-shifting operator that adapts to the given problem in some way. Bienstock and Zuckerberg [BZ04] devised the first operators that generate different variables for different relaxations (or even different algebraic descriptions of the same relaxation). They showed that this flexibility can be very useful in attacking relaxations of some set covering problems. Thus, perhaps tight relaxations for other hard problems can be found similarly by building a lift-and-project operator with suitable adaptations.

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