STABLE SET POLYTOPES WITH HIGH LIFT-AND-PROJECT RANKS FOR THE LOVÁSZ–SCHRIJVER SDP OPERATOR

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Abstract. We study the lift-and-project rank of the stable set polytopes of graphs with respect to the Lovász–Schrijver SDP operator LS_+, with a particular focus on a search for relatively small graphs with high LS_+-rank (the least number of iterations of the LS_+ operator on the fractional stable set polytope to compute the stable set polytope). We provide families of graphs whose LS_+-rank is asymptotically a linear function of its number of vertices, which is the best possible up to improvements in the constant factor (previous best result in this direction, from 1999, yielded graphs whose LS_+-rank only grew with the square root of the number of vertices).

1. Introduction

In combinatorial optimization, a standard approach for tackling a given problem is to encode its set of feasible solutions geometrically (e.g., via an integer programming formulation). While the exact solution set is often difficult to analyze, we can focus on relaxations of this set that have certain desirable properties (e.g., combinatorially simple to describe, approximates the underlying set of solutions well, and/or is computationally efficient to optimize over). In that regard, the *lift-and-project* approach provides a systematic procedure which generates progressively tighter convex relaxations of any given $0,1$ optimization problem. In the last four decades, many procedures that fall under the lift-and-project approach have been devised (see, among others, [SA90, LS91, BCC93, Las01, BZ04, AT16]), and there is an extensive body of work on their general properties and performance on a wide range of discrete optimization problems (see, for instance, [Au14] and the references therein). Lift-and-project operators can be classified into two groups based on the type of convex relaxations they generate: Those that generate polyhedral relaxations only (leading to Linear Programming relaxations), and those that generate spectrahedral relaxations (leading to Semidefinite Programming relaxations) which are not necessarily polyhedral. Herein, we focus on LS_+ (defined in detail in Section 2), the SDP-based lift-and-project operator due to Lovász and Schrijver [LS91], and its performance on the stable set problem of graphs. This lift-and-project operator was originally called N_+ in [LS91].

A remarkable property of LS_+ is that applying one iteration of the operator to the fractional stable set polytope of a graph already yields a tractable relaxation of the stable set polytope that is stronger than the Lovász theta body relaxation (see [LS91, Lemma 2.17 and Corollary 2.18]). Thus, LS_+ has been shown to perform well on the stable set problem for graphs that are perfect or "close" to being perfect [BENT13, BENT17, Wag22, BENW23]. For more analyses of

Date: January 2, 2024.

Key words and phrases. stable set problem, lift and project, combinatorial optimization, semidefinite programming, integer programming.

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various lift-and-project relaxations of the stable set problem, see (among others) [LS91, dKP02, Lau03, LT03, BO04, EMN06, GL07, PnVZ07, GLRS09, GRS13, AT16, ALT22].

LS_+ is an operator on convex subsets of the hypercube $[0,1]^n$ — given a convex set $P \subseteq [0,1]^n$, $\text{LS}_+(P)$ is a convex set sandwiched between P and the convex hull of integral points in P, denoted by P_I:

$$P \supseteq \text{LS}_+(P) \supseteq P_I.$$

The convex hull of integral points in a set is also called its *integer hull*. So, P_I is the integer hull of P.

We can apply LS_+ many times to obtain yet tighter relaxations. Given $p \in \mathbb{N}$, let $\text{LS}_+^p(P)$ be the set obtained from applying k successive LS_+ operations to P. Then it is well known (e.g., it follows from [LS91, Theorem 1.4] and the definition of LS_+) that

$$\text{LS}_+^0(P) := P \supseteq \text{LS}_+(P) \supseteq \text{LS}_+^2(P) \supseteq \cdots \supseteq \text{LS}_+^{n-1}(P) \supseteq \text{LS}_+^n(P) = P_I.$$

Thus, LS_+ generates a hierarchy of progressively tighter convex relaxations which converge to P_I in no more than n iterations. The reader may refer to Lovász and Schrijver [LS91] for some other fundamental properties of the LS_+ operator.

The LS_+*-rank* of a convex subset of the hypercube is defined to be the number of iterations it takes LS_+ to return its integer hull. As we stated above, the LS_+-rank of every convex set $P \subseteq [0,1]^n$ is at most n, and a number of elementary polytopes in $\mathbb{R}^n$ have been shown to have LS_+-rank $\Theta(n)$ (see, among others, [Goe98, CD01, GT01, STT07, AT18]).

On the other hand, while the stable set problem is known to be strongly $\mathcal{NP}$-hard, hardness results for LS_+ (or any other lift-and-project operator utilizing semidefinite programming) on the stable set problem have been relatively scarce. Given a graph $G = (V, E)$, we denote by $\text{STAB}(G)$ the stable set polytope of G (the convex hull of incidence vectors of stable sets in G), and by $\text{FRAC}(G)$ the fractional stable set polytope of G (polytope defined by edge inequalities and $x \in [0,1]^n$, see Section 2.2). Then, we define the LS_+-rank of a graph G as the minimum non-negative integer p for which $\text{LS}_+^p(\text{FRAC}(G)) = \text{STAB}(G)$. We denote the LS_+-rank of a graph G by $r_+(G)$. Since we mentioned that $\text{LS}_+(G)$ is a subset of the Lovász theta body of G, we already know that for every perfect graph G, $r_+(G) \leq 1$ (and for every bipartite graph G, in which case $\text{FRAC}(G) = \text{STAB}(G)$, $r_+(G) = 0$).

Prior to this manuscript, the worst (in terms of performance by LS_+) family of graphs was given by the line graphs of odd cliques. Using the fact that the LS_+-rank of the fractional matching polytope of the $(2k+1)$-clique is k [ST99], the natural correspondence between the matchings in a given graph and the stable sets in the corresponding line graph, as well as the properties of the LS_+ operator, it follows that the fractional stable set polytope of the line graph of the $(2k+1)$-clique (which contains $\binom{2k+1}{2} = k(2k+1)$ vertices) has LS_+-rank k, giving a family of graphs G with $r_+(G) = \Theta\left(\sqrt{|V(G)|}\right)$. This lower bound on the LS_+-rank of the fractional stable set polytopes has not been improved since 1999. In this manuscript, we present what we believe is the first known family of graphs whose LS_+-rank is asymptotically a linear function of the number of vertices.

1.1. A family of challenging graphs $\{H_k\}$ for the LS_+ operator.

For a positive integer k, let $[k] := \{1, 2, \ldots, k\}$. We first define the family of graphs that pertains to our main result.

Definition 1. Given an integer $k \geq 2$, define H_k to be the graph where

$$V(H_k) := \{i_p : i \in [k], p \in \{0,1,2\}\},$$

and the edges of H_k are

- $\{i_0, i_1\}$ and $\{i_1, i_2\}$ for every $i \in [k]$;

- $\{i_0, j_2\}$ for all $i, j \in [k]$ where $i \neq j$.

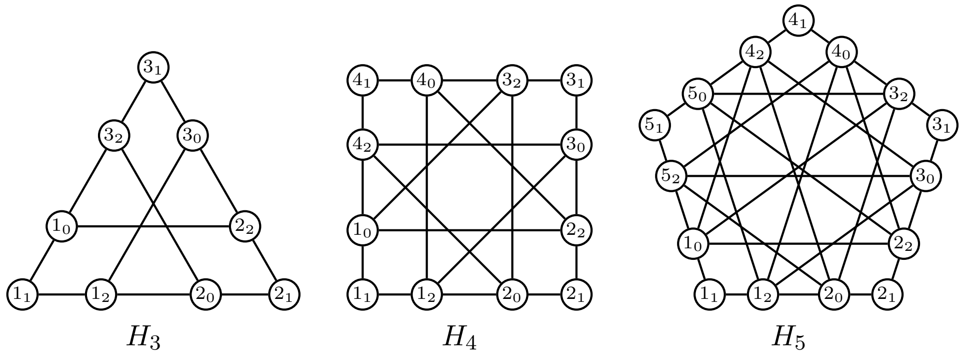

FIGURE 1. Several graphs in the family H_k

Figure 1 illustrates the graphs H_k for several small values of k. Note that H_2 is the cycle on six vertices. Moreover, H_k can be obtained from a complete bipartite graph by fixing a perfect matching and replacing each edge by a path of length two (subdividing each matching edge). Figure 2 represents a drawing emphasizing this second feature.

The following is the main result of this paper. Its proof is given in Section 4.

Theorem 2. *For every $k \geq 3$, the* LS$_+$*-rank of the fractional stable set polytope of H_k is at least $\frac{1}{16}|V(H_k)|$.*

We remark that, given $p \in \mathbb{N}$ and polytope $P \subseteq [0,1]^n$ with $\Omega(n^c)$ facets for some constant c, the straightforward formulation of $\text{LS}_+^p(P)$ is an SDP with size $n^{\Omega(p)}$. Since the fractional stable set polytope of the graph H_k has dimension $n = 3k$ with $\Omega(n^2)$ facets, Theorem 2 implies that the SDP described by LS$_+$ that fails to exactly represent the stable set polytope of H_k has size $n^{\Omega(n)}$. Thus, the consequence of Theorem 2 on the size of the SDPs generated from the LS$_+$ operator is incomparable with the extension complexity bound due to Lee, Raghavendra, and Steurer [LRS15], who showed (in the context of SDP extension complexity) that it takes an SDP of size $2^{\Omega(n^{1/13})}$ to exactly represent the stable set polytope of a general n-vertex graph as a projection of a spectrahedron. That is, while the general lower bound by [LRS15] covers every lifted-SDP formulation, our lower bound is significantly larger but it only applies to a specific family of lifted-SDP formulations.

1.2. **Organization of the paper.** In Section 2, we introduce the LS$_+$ operator and the stable set problem, and establish some notation and basic facts that will aid our subsequent discussion. In Section 3, we study the family of graphs H_k, their stable set polytope and set up some fundamental facts and the proof strategy. We then prove Theorem 2 in Section 4. In Section 5 we determine the Chvátal–Gomory rank of the stable set polytope of graphs H_k. In Section 6, we show that the results in Sections 3 and 4 readily lead to the discovery of families of vertex-transitive graphs whose LS$_+$-rank also exhibits asymptotically linear growth. Finally, in Section 7, we close by mentioning some natural research directions inspired by these new findings.

2. Preliminaries

In this section, we establish the necessary definitions and notation for our subsequent analysis.

2.1. **The lift-and-project operator** LS_+**.** Here, we define the lift-and-project operator LS_+ due to Lovász and Schrijver [LS91] and mention some of its basic properties. Given a convex set $P \subseteq [0,1]^n$, we define the cone

$$\text{cone}(P) := \left\{ \begin{bmatrix} \lambda \\ \lambda x \end{bmatrix} : \lambda \geq 0, x \in P \right\},$$

and index the new coordinate by 0. Given a vector x and an index i, we may refer to the i-entry in x by x_i or $[x]_i$. All vectors are column vectors, so here the transpose of x, $x^\top$, is a row vector. Next, let $\mathbb{S}_+^n$ denote the set of n-by-n symmetric positive semidefinite matrices, and $\text{diag}(Y)$ be the vector formed by the diagonal entries of a square matrix Y. We also let e_i be the i^{th} unit vector.

Given $P \subseteq [0,1]^n$, the operator LS_+ first *lifts* P to the following set of matrices:

$$\widehat{\text{LS}}_+(P) := \left\{ Y \in \mathbb{S}_+^{n+1} : Ye_0 = \text{diag}(Y), Ye_i, Y(e_0 - e_i) \in \text{cone}(P)\ \forall i \in [n] \right\}.$$

It then *projects* the set back down to the following set in $\mathbb{R}^n$:

$$\text{LS}_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\text{LS}}_+(P), Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix} \right\}.$$

The following is a foundational property of LS_+.

Lemma 3. *Let $P \subseteq [0,1]^n$ be a convex set. Then,*

$$P \supseteq \text{LS}_+(P) \supseteq P_I.$$

Proof. Given $x \in \text{LS}_+(P)$, let $Y \in \widehat{\text{LS}}_+(P)$ be the corresponding *certificate matrix* (and so $Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix}$). Since $Ye_0 = Ye_i + Y(e_0 - e_i)$ for any index $i \in [n]$ and that $\widehat{\text{LS}}_+$ imposes that $Ye_i, Y(e_0 - e_i) \in \text{cone}(P)$, it follows that $Ye_0 \in \text{cone}(P)$, and thus $x \in P$. On the other hand, given any integral vector $x \in P \cap \{0,1\}^n$, observe that $Y := \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^\top \in \widehat{\text{LS}}_+(P)$, and so $x \in \text{LS}_+(P)$. Thus, since $P_I := \text{conv}\left(P \cap \{0,1\}^n\right)$ (i.e., the integer hull of P), we deduce

$$P_I \subseteq \text{LS}_+(P) \subseteq P$$

holds. □

Therefore, $\text{LS}_+(P)$ contains the same set of integral solutions as P. Moreover, if P is a tractable set (i.e., one can optimize a linear function over P in polynomial time), then so is $\text{LS}_+(P)$. It is also known that $\text{LS}_+(P)$ is strictly contained in P unless $P = P_I$. Thus, while it is generally $\mathcal{NP}$-hard to optimize over the integer hull P_I, $\text{LS}_+(P)$ offers a tractable relaxation of P_I that is tighter than the initial relaxation P.

2.2. **The stable set polytope and the** LS_+**-rank of graphs.** Given a graph $G := (V(G), E(G))$, we define its *fractional stable set polytope* to be

$$\text{FRAC}(G) := \left\{ x \in [0,1]^{V(G)} : x_i + x_j \leq 1, \forall \{i,j\} \in E(G) \right\}.$$

We also define

$$\text{STAB}(G) := \text{FRAC}(G)_I = \text{conv}\left(\text{FRAC}(G) \cap \{0,1\}^{V(G)}\right)$$

to be the *stable set polytope* of G. Notice that $\text{STAB}(G)$ is exactly the convex hull of the incidence vectors of stable sets in G. Also, to reduce cluttering, we will write $\text{LS}_+^p(G)$ instead of $\text{LS}_+^p(\text{FRAC}(G))$.

Given a graph G, recall that we let $r_+(G)$ denote the LS$_+$-*rank* of G, which is defined to be the smallest integer p where $\text{LS}_+^p(G) = \text{STAB}(G)$. More generally, given a linear inequality $a^\top x \leq \beta$ valid for STAB(G), we define its LS$_+$-*rank* to be the smallest integer p for which $a^\top x \leq \beta$ is a valid inequality for $\text{LS}_+^p(G)$. Then $r_+(G)$ can be alternatively defined as the maximum LS$_+$-rank over all valid inequalities of STAB(G).

It is well known that $r_+(G) = 0$ (i.e., $\text{STAB}(G) = \text{FRAC}(G)$) if and only if G is bipartite. There has also been recent interest and progress in classifying graphs with $r_+(G) = 1$, which are commonly called LS$_+$-*perfect graphs* [BENT13, BENT17, Wag22, BENW23]. An important family of LS$_+$-perfect graphs are the *perfect graphs*, which are graphs whose stable set polytope is defined by only clique and non-negativity inequalities. (See [LS91, Lemma 1.5] for a proof that clique inequalities are valid for LS$_+(G)$.) In addition to perfect graphs, odd holes, odd antiholes, and odd wheels are also known to have LS$_+$-rank 1. It remains an open problem to find a simple combinatorial characterization of LS$_+$-perfect graphs.

Next, we mention two simple graph operations that have been critical to the analyses of the LS$_+$-ranks of graphs. Given a graph G and $S \subseteq V(G)$, we let $G - S$ denote the subgraph of G induced by the vertices $V(G) \setminus S$, and call $G - S$ the graph obtained by the *deletion* of S. (When $S = \{i\}$ for some vertex i, we simply write $G - i$ instead of $G - \{i\}$.) Next, given $i \in V(G)$, let $\Gamma(i) \coloneqq \{j \in V(G) : \{i, j\} \in E(G)\}$ (i.e., $\Gamma(i)$ is the set of vertices that are adjacent to i). Then the graph obtained from the *destruction* of i in G is defined as

$$G \ominus i \coloneqq G - (\{i\} \cup \Gamma(i)).$$

Then we have the following.

Theorem 4. *For every graph G,*

(i) [LS91, Corollary 2.16] $r_+(G) \leq \max\{r_+(G \ominus i) : i \in V(G)\} + 1$*;*
(ii) [LT03, Theorem 36] $r_+(G) \leq \min\{r_+(G - i) : i \in V(G)\} + 1$.

The following is another elementary property of LS$_+$ that will be useful.

Lemma 5. *Let G be a graph, and let G' be an induced subgraph of G. Then $r_+(G') \leq r_+(G)$.*

Proof. First, there is nothing to prove if $r_+(G') = 0$, so we assume that $r_+(G') = p + 1$ for some $p \geq 0$. Thus, there exists $\bar{x}' \in \text{LS}_+^p(G') \setminus \text{STAB}(G')$.

Now define $\bar{x} \in \mathbb{R}^{V(G)}$ where $\bar{x}_i = \bar{x}'_i$ if $i \in V(G')$ and $\bar{x}_i = 0$ otherwise. It is easy to see that $\bar{x}' \notin \text{STAB}(G') \Rightarrow \bar{x} \notin \text{STAB}(G)$. Next, it is known that for every face $F \subseteq [0, 1]^n$, convex set $P \subseteq [0, 1]^n$, and integer $p \geq 0$,

$$\text{LS}_+^p(P \cap F) = \text{LS}_+^p(P) \cap F.$$

Applying the above with $P \coloneqq \text{FRAC}(G)$ and $F \coloneqq \left\{x \in [0,1]^{V(G)} : x_i = 0 \ \forall i \in V(G) \setminus V(G')\right\}$, we obtain that $\bar{x} \in \text{LS}_+^p(G)$. Thus, we conclude that $\bar{x} \in \text{LS}_+^p(G) \setminus \text{STAB}(G)$, and $r_+(G) \geq p + 1$. $\square$

We are interested in studying relatively small graphs with high LS$_+$-rank — that is, graphs whose stable set polytope is difficult to obtain for LS$_+$. First, Lipták and the second author [LT03, Theorem 39] proved the following general upper bound:

Theorem 6. *For every graph G, $r_+(G) \leq \left\lfloor \frac{|V(G)|}{3} \right\rfloor$.*

In Section 4, we prove that the family of graphs H_k satisfies $r_+(H_k) = \Theta(|V(H_k)|)$. This shows that Theorem 6 is asymptotically tight, and rules out the possibility of a sublinear upper bound on the LS$_+$-rank of a general graph.

3. Analyzing H_k and exploiting symmetries

3.1. **The graphs H_k and their basic properties.** Recall the family of graphs H_k defined in Section 1 (Definition 1). For convenience, we let $[k]_p := \{j_p : j \in [k]\}$ for each $p \in \{0, 1, 2\}$. Then, as mentioned earlier, one can construct H_k by starting with a complete bipartite graph with bipartitions $[k]_0$ and $[k]_2$, and then for every $j \in [k]$ subdividing the edge $\{j_0, j_2\}$ into a path of length 2 and labelling the new vertex j_1. Figure 2 illustrates alternative drawings for H_k which highlight this aspect of the family of graphs.

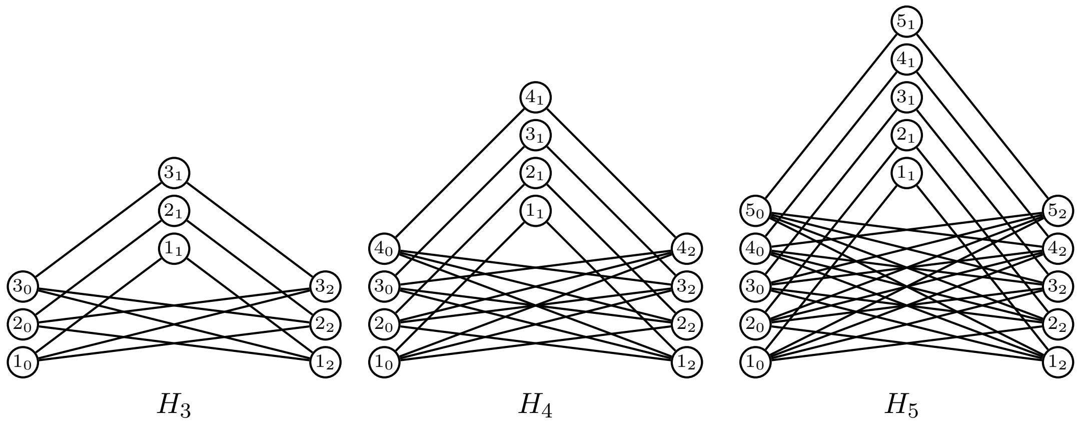

Figure 2. Alternative drawings of the graphs H_k

Given a graph G, we say that $\sigma : V(G) \to V(G)$ is an *automorphism* of G if, for every $i, j \in V(G)$, $\{i, j\} \in E(G)$ if and only if $\{\sigma(i), \sigma(j)\} \in E(G)$. Notice that the graphs H_k have very rich symmetries, and we mention two automorphisms of H_k that are of particular interest. Define $\sigma_1 : V(H_k) \to V(H_k)$ where

$$\sigma_1(j_p) := \begin{cases} (j+1)_p & \text{if } 1 \leq j \leq k-1; \\ 1_p & \text{if } j = k. \end{cases} \tag{1}$$

Also define $\sigma_2 : V(H_k) \to V(H_k)$ where

$$\sigma_2(j_p) := j_{(2-p)} \text{ for every } j \in [k] \text{ and } p \in \{0, 1, 2\}. \tag{2}$$

Visually, σ_1 corresponds to rotating the drawings of H_k in Figure 1 counterclockwise by $\frac{2\pi}{k}$, and σ_2 corresponds to reflecting the drawings of H_k in Figure 2 along the centre vertical line. Also, notice that $H_k \ominus i$ is either isomorphic to H_{k-1} (if $i \in [k]_1$), or is bipartite (if $i \in [k]_0 \cup [k]_2$). As we shall see, these properties are very desirable in our subsequent analysis of H_k.

Given a graph G, let $\alpha(G)$ denote the cardinality of the maximum stable set in a graph G. Notice that since H_2 is the 6-cycle, $\alpha(H_2) = 3$, and the maximum cardinality stable sets of H_2 are $\{1_0, 1_2, 2_1\}$ and $\{1_1, 2_0, 2_2\}$. Moreover, due to the simple recursive structure of this family of graphs, we can construct stable sets for H_k from stable sets for H_{k-1} for every integer $k \geq 3$. If S is a (maximum cardinality) stable set for H_{k-1} then $S \cup \{k_1\}$ is a (maximum cardinality) stable set for H_k. This shows that $\alpha(H_k) = k + 1$ for every $k \geq 2$.

Also, notice that each of the sets $[k]_0$, $[k]_1$ and $[k]_2$ is a stable set in H_k. While they each have cardinality k and thus are not maximum cardinality stable sets in H_k, they are inclusion-wise maximal (i.e., each of them is not a proper subset of another stable set in H_k). The following result characterizes all inclusion-wise maximal stable sets in H_k.

Lemma 7. *Let $k \geq 2$ and let $S \subseteq V(H_k)$ be an inclusion-wise maximal stable set in H_k. Then one of the following is true:*

(i) $|S| = k$, $|S \cap \{j_0, j_1, j_2\}| = 1$ *for every* $j \in [k]$, *and either* $S \cap [k]_0 = \emptyset$ *or* $S \cap [k]_2 = \emptyset$*;*

(ii) $|S| = k + 1$, *and there exists* $j \in [k]$ *where*

$$S = ([k]_1 \setminus \{j_1\}) \cup \{j_0, j_2\}.$$

Proof. First, notice that if $|S \cap \{j_0, j_1, j_2\}| = 0$ for some $j \in [k]$, then $S \cup \{j_1\}$ is a stable set, contradicting the maximality of S. Thus, we assume that $|S \cap \{j_0, j_1, j_2\}| \geq 1$ for every $j \in [k]$.

If $|S \cap \{j_0, j_1, j_2\}| = 1$ for all $j \in [k]$, then $|S| = k$. Also, since $\{j_0, j'_2\}$ is an edge for every distinct $j, j' \in [k]$, we see that $S \cap [k]_0$ and $S \cap [k]_2$ cannot both be non-empty. Thus, S belongs to (i) in this case.

Next, suppose there exists $j \in [k]$ where $|S \cap \{j_0, j_1, j_2\}| \geq 2$. Then it must be that $j_0, j_2 \in S$ and $j_1 \notin S$. Then it follows that, for all $j' \neq j$, $S \cap \{j'_0, j'_1, j'_2\} = \{j'_1\}$, and S belongs to (ii). □

Next, we describe two families of valid inequalities of $\text{STAB}(H_k)$ that are of particular interest. Given distinct indices $j, j' \in [k]$, define

$$B_{j,j'} := V(H_k) \setminus \{j_1, j_2, j'_0, j'_1\}.$$

Then we have the following.

Lemma 8. *For every integer $k \geq 2$,*

(i) *the linear inequality*

$$\sum_{i \in B_{j,j'}} x_i \leq k - 1 \tag{3}$$

is a facet of $\text{STAB}(H_k)$ *for every pair of distinct* $j, j' \in [k]$.

(ii) *the linear inequality*

$$\sum_{i \in [k]_0 \cup [k]_2} (k-1)x_i + \sum_{i \in [k]_1} (k-2)x_i \leq k(k-1) \tag{4}$$

is valid for $\text{STAB}(H_k)$.

Proof. We first prove (i) by induction on k. When $k = 2$, (3) gives an edge inequality, which is indeed a facet of $\text{STAB}(H_2)$ since H_2 is the 6-cycle.

Next, assume $k \geq 3$. By the symmetry of H_k, it suffices to prove the claim for the case $j = 1$ and $j' = 2$. First, it follows from Lemma 7 that $B_{1,2}$ does not contain a stable set of size k, and so (3) is valid for $\text{STAB}(H_k)$. Next, by the inductive hypothesis,

$$\left(\sum_{i \in B_{1,2}} x_i \right) - (x_{k_0} + x_{k_1} + x_{k_2}) \leq k - 2 \tag{5}$$

is a facet of $\text{STAB}(H_{k-1})$, and so there exist stable sets $S_1, \ldots, S_{3k-3} \subseteq V(H_{k-1})$ whose incidence vectors are affinely independent and all satisfy (5) with equality. We then define $S'_i := S_i \cup \{k_1\}$ for all $i \in [3k-3]$, $S'_{3k-2} := [k]_0, S'_{3k-1} := [k]_2$, and $S'_{3k} := [k-1]_1 \cup \{k_0, k_2\}$. Then we see that the incidence vectors of $S'_1, \ldots, S'_{3k}$ are affinely independent, and they all satisfy (3) with equality. This finishes the proof of (i).

We next prove (ii). Consider the inequality obtained by summing (3) over all distinct $j, j' \in [k]$:

$$\sum_{(j,j') \in [k]^2, j \neq j'} \left(\sum_{i \in B_{j,j'}} x_i \right) \leq \sum_{(j,j') \in [k]^2, j \neq j'} (k-1). \tag{6}$$

Now, the right hand side of (6) is $k(k-1)(k-1)$. On the other hand, since $|B_{j,j'} \cap [k]_0| = |B_{j,j'} \cap [k]_2| = k-1$ for all j, j', we see that if $i \in [k]_0 \cup [k]_2$, then x_i has coefficient $(k-1)(k-1)$ in the left hand side of (6). A similar argument shows that x_i has coefficient $(k-1)(k-2)$ for all $i \in [k]_1$. Thus, (4) is indeed $\frac{1}{k-1}$ times (6). Therefore, (4) is a non-negative linear combination of inequalities of the form (3), so it follows from (i) that (4) is valid for $\mathrm{STAB}(H_k)$. ☐

3.2. **Working from the shadows to prove lower bounds on LS_+-rank.** Next, we aim to exploit the symmetries of H_k to help simplify our analysis of its LS_+-relaxations. Before we do that, we describe the broader framework of this reduction that shall also be useful in analyzing lift-and-project relaxations in other settings. Given a graph G, let $\mathrm{Aut}(G)$ denote the automorphism group of G. We also let χ_S denote the incidence vector of a set S. Then we define the notion of $\mathcal{A}$-balancing automorphisms.

Definition 9. Given a graph G and $\mathcal{A} \coloneqq \{A_1, \ldots, A_L\}$ a partition of $V(G)$, we say that a set of automorphisms $\mathcal{S} \subseteq \mathrm{Aut}(G)$ is $\mathcal{A}$-*balancing* if

(1) For every $\ell \in [L]$, and for every $\sigma \in \mathcal{S}$, $\{\sigma(i) : i \in A_\ell\} = A_\ell$.
(2) For every $\ell \in [L]$, the quantity $|\{\sigma \in S : \sigma(i) = j\}|$ is invariant under the choice of $i, j \in A_\ell$.

In other words, if $\mathcal{S}$ is $\mathcal{A}$-balancing, then the automorphisms in $\mathcal{S}$ only map vertices in A_ℓ to vertices in A_ℓ for every $\ell \in [L]$. Moreover, for every $i \in A_\ell$, the $|\mathcal{S}|$ images of i under automorphisms in $\mathcal{S}$ spread over A_ℓ evenly, with $|\{\sigma \in \mathcal{S} : \sigma(i) = j\}| = \frac{|\mathcal{S}|}{|A_\ell|}$ for every $j \in A_\ell$.

For example, for the graph H_k, consider the vertex partition $\mathcal{A}_1 \coloneqq \{[k]_0, [k]_1, [k]_2\}$ and $\mathcal{S}_1 \coloneqq \left\{\sigma_1^j : j \in [k]\right\}$ (where σ_1 is as defined in (1)). Then observe that, for every $p \in \{0, 1, 2\}$, $\{\sigma(i) : i \in [k]_p\} = [k]_p$, and

$$|\{\sigma \in \mathcal{S}_1 : \sigma(i) = j\}| = 1$$

for every $i, j \in [k]_p$. Thus, $\mathcal{S}_1$ is $\mathcal{A}_1$-balancing. Furthermore, if we define $\mathcal{A}_2 \coloneqq \{[k]_0 \cup [k]_2, [k]_1\}$ and

$$\mathcal{S}_2 \coloneqq \left\{\sigma_j^i \circ \sigma_2^{j'} : j \in [k], j' \in [2]\right\} \tag{7}$$

(where σ_2 is as defined in (2)), one can similarly show that $\mathcal{S}_2$ is $\mathcal{A}_2$-balancing.

Next, we prove several lemmas about $\mathcal{A}$-balancing automorphisms that are relevant to the analysis of LS_+-relaxations. Given $\sigma \in \mathrm{Aut}(G)$, we extend the notation to refer to the function $\sigma : \mathbb{R}^{V(G)} \to \mathbb{R}^{V(G)}$ where $\sigma(x)_i = x_{\sigma(i)}$ for every $i \in V(G)$. The following follows readily from the definition of $\mathcal{A}$-balancing automorphisms.

Lemma 10. *Let G be a graph, $\mathcal{A} \coloneqq \{A_1, \ldots, A_L\}$ be a partition of $V(G)$, and $\mathcal{S} \subseteq \mathrm{Aut}(G)$ be a set of $\mathcal{A}$-balancing automorphisms. Then, for every $x \in \mathbb{R}^{V(G)}$,*

$$\frac{1}{|\mathcal{S}|} \sum_{\sigma \in \mathcal{S}} \sigma(x) = \sum_{\ell=1}^{L} \frac{1}{|A_\ell|} \left(\sum_{i \in A_\ell} x_i \right) \chi_{A_\ell}.$$

Proof. For every $\ell \in [L]$ and for every $i \in A_\ell$, the fact that $\mathcal{S}$ is $\mathcal{A}$-balancing implies

$$\frac{1}{|\mathcal{S}|}\sum_{\sigma \in \mathcal{S}} \sigma(e_i) = \frac{1}{|A_\ell|}\chi_{A_\ell}. \tag{8}$$

Since $x = \sum_{i \in V(G)} x_i e_i$, the claim follows by summing x_i times (8) over all $i \in V(G)$. □

Lemma 11. *Let G be a graph, $\sigma \in \text{Aut}(G)$ be an automorphism of G, and $p \geq 0$ be an integer. If $x \in \text{LS}_+^p(G)$, then $\sigma(x) \in \text{LS}_+^p(G)$.*

Proof. When $p = 0$, $\text{LS}_+^0(G) = \text{FRAC}(G)$, and the claim holds due to σ being an automorphism of G. Next, it is easy to see from the definition of LS_+ that the operator is invariant under permutation of coordinates (i.e., given $P \subseteq [0,1]^n$, if we let $P_\sigma \coloneqq \{\sigma(x) : x \in P\}$, then $x \in \text{LS}_+(P) \Rightarrow \sigma(x) \in \text{LS}_+(P_\sigma)$). Applying this insight recursively proves the claim for all p. □

Combining the results from the above, we obtain the following.

Proposition 12. *Suppose G is a graph, $\mathcal{A} \coloneqq \{A_1, \ldots, A_L\}$ is a partition of $V(G)$, and $\mathcal{S} \subseteq \text{Aut}(G)$ is $\mathcal{A}$-balancing. Let $p \geq 0$ be an integer. If $x \in \text{LS}_+^p(G)$, then*

$$x' \coloneqq \sum_{\ell=1}^{L} \left(\frac{1}{|A_\ell|} \sum_{i \in A_\ell} x_i \right) \chi_{A_\ell}$$

also belongs to $\text{LS}_+^p(G)$.

Proof. Since $\mathcal{S}$ is $\mathcal{A}$-balancing, it follows from Lemma 10 that

$$x' = \frac{1}{|\mathcal{S}|}\sum_{\sigma \in \mathcal{S}} \sigma(x).$$

Also, since $x \in \text{LS}_+^p(G)$, Lemma 11 implies that $\sigma(x) \in \text{LS}_+^p(G)$ for every $\sigma \in \mathcal{S}$. Thus, x' is a convex combination of points in $\text{LS}_+^p(G)$, which is a convex set. Hence, it follows that $x' \in \text{LS}_+^p(G)$. □

Thus, the presence of $\mathcal{A}$-balancing automorphisms allows us to focus on points in $\text{LS}_+^p(G)$ with fewer distinct entries. That is, instead of fully analyzing a family of SDPs in $\mathbb{S}_+^{\Omega(n^p)}$ or its projections $\text{LS}_+^p(G)$, we can work with a spectrahedral shadow in $[0,1]^L$ for a part of the analysis. For instance, in the extreme case when G is vertex-transitive, we see that the entire automorphism group $\text{Aut}(G)$ is $\{V(G)\}$-balancing, and so for every $x \in \text{LS}_+^p(G)$, Proposition 12 implies that $\frac{1}{|V(G)|}\left(\sum_{i \in V(G)} x_i\right) \bar{e} \in \text{LS}_+^p(G)$, where $\bar{e}$ denotes the vector of all ones.

Now we turn our focus back to the graphs H_k. The presence of an $\mathcal{A}_2$-balancing set of automorphisms (as described in (7)) motivates the study of points in $\text{LS}_+^p(H_k)$ of the following form.

Definition 13. Given real numbers $a, b \in \mathbb{R}$ and integer $k \geq 2$, the vector $w_k(a,b) \in \mathbb{R}^{V(H_k)}$ is defined such that

$$[w_k(a,b)]_i \coloneqq \begin{cases} a & \text{if } i \in [k]_0 \cup [k]_2; \\ b & \text{if } i \in [k]_1. \end{cases}$$

For an example, the inequality (4) can be rewritten as $w(k-1, k-2)^\top x \leq k(k-1)$. The following is a main reason why we are interested in looking into points of the form $w_k(a,b)$.

Lemma 14. *Suppose there exists $x \in \text{LS}_+^p(H_k)$ where x violates (4). Then there exist real numbers a, b where $w_k(a,b) \in \text{LS}_+^p(H_k)$ and $w_k(a,b)$ violates (4).*

Proof. Given x, let $a := \frac{1}{2k}\sum_{i\in[k]_0\cup[k]_2} x_i$ and $b := \frac{1}{k}\sum_{i\in[k]_1} x_i$. Due to the presence of the $\mathcal{A}_2$-balancing automorphisms $\mathcal{S}_2$, as well as Proposition 12, we know that $x' := w_k(a,b)$ belongs to $\mathrm{LS}_+^p(H_k)$. Now since x violates (4),

$$k(k-1) < w(k-1,k-2)^\top x = (k-1)(2ka) + (k-2)(kb) = w(k-1,k-2)^\top x',$$

and so x' violates (4) as well. □

A key ingredient of our proof of the main result is to find a point $x \in \mathrm{LS}_+^p(H_k)$ where x violates (4) for some $p \in \Theta(k)$, which would imply that $r_+(H_k) > p$. Lemma 14 assures that, due to the symmetries of H_k, we are not sacrificing any sharpness of the result by only looking for such points x of the form $w_k(a,b)$. This enables us to capture important properties of $\mathrm{LS}_+^p(H_k)$ by analyzing a corresponding "shadow" of the set in $\mathbb{R}^2$. More explicitly, given $P \subseteq \mathbb{R}^{V(H_k)}$, we define

$$\Phi(P) := \left\{(a,b) \in \mathbb{R}^2 : w_k(a,b) \in P\right\}.$$

For example, it is not hard to see that

$$\Phi(\mathrm{FRAC}(H_k)) = \mathrm{conv}\left(\left\{(0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right), (0,1)\right\}\right)$$

for every $k \geq 2$. We can similarly characterize $\Phi(\mathrm{STAB}(H_k))$.

Lemma 15. *For every integer $k \geq 2$, we have*

$$\Phi(\mathrm{STAB}(H_k)) = \mathrm{conv}\left(\left\{(0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{k},\frac{k-1}{k}\right), (0,1)\right\}\right).$$

Proof. Let $k \geq 2$ be an integer. Then, the empty set, $[k]_1$, $[k]_0$, and $[k]_2$ are all stable sets in H_k. Notice that $\chi_\emptyset = w_k(0,0)$ and $\chi_{[k]_1} = w_k(0,1)$, and thus $(0,0)$ and $(0,1)$ are both in $\Phi(\mathrm{STAB}(H_k))$. Also, since $\frac{1}{2}\chi_{[k]_0} + \frac{1}{2}\chi_{[k]_2} = w_k\left(\frac{1}{2},0\right) \in \mathrm{STAB}(H_k)$, we have $\left(\frac{1}{2},0\right) \in \Phi(\mathrm{STAB}(H_k))$. Next, recall from Lemma 7 that for every $j \in [k]$,

$$S_j := ([k]_1 \setminus \{j_1\}) \cup \{j_0, j_2\}$$

is a stable set of H_k. Thus, $\frac{1}{k}\sum_{j=1}^k \chi_{S_j} = w_k\left(\frac{1}{k},\frac{k-1}{k}\right) \in \mathrm{STAB}(H_k)$, and so $\left(\frac{1}{k},\frac{k-1}{k}\right) \in \Phi(\mathrm{STAB}(H_k))$. Therefore, $\Phi(\mathrm{STAB}(H_k)) \supseteq \mathrm{conv}\left(\left\{(0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{k},\frac{k-1}{k}\right), (0,1)\right\}\right)$.

On the other hand, for all $(a,b) \in \Phi(\mathrm{STAB}(H_k))$, it follows from Lemma 8 that

$$2(k-1)a + (k-2)b \leq k-1$$

Since $\Phi(\mathrm{STAB}(H_k)) \subseteq \Phi(\mathrm{FRAC}(H_k))$, using our characterization of $\Phi(\mathrm{FRAC}(H_k))$, we deduce $\Phi(\mathrm{STAB}(H_k))$ is contained in the set

$$\mathrm{conv}\left(\left\{(0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right), (0,1)\right\}\right) \cap \left\{(a,b) \in \mathbb{R}^2 \,:\, 2(k-1)a + (k-2)b \leq k-1\right\}.$$

However, the above set is exactly $\mathrm{conv}\left(\left\{(0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{k},\frac{k-1}{k}\right), (0,1)\right\}\right)$. Therefore,

$$\Phi(\mathrm{STAB}(H_k)) = \mathrm{conv}\left(\left\{(0,0), \left(\frac{1}{2},0\right), \left(\frac{1}{k},\frac{k-1}{k}\right), (0,1)\right\}\right).$$

□

Even though $\mathrm{STAB}(H_k)$ is an integral polytope, notice that $\Phi(\mathrm{STAB}(H_k))$ is not integral. Nonetheless, it is clear that

$$\mathrm{LS}_+^p(H_k) = \mathrm{STAB}(H_k) \Rightarrow \Phi(\mathrm{LS}_+^p(H_k)) = \Phi(\mathrm{STAB}(H_k)).$$

Thus, to show that $r_+(H_k) > p$, it suffices to find a point $(a,b) \in \Phi(\mathrm{LS}_+^p(H_k)) \setminus \Phi(\mathrm{STAB}(H_k))$. More generally, given a graph G with a set of $\mathcal{A}$-balancing automorphisms where $\mathcal{A}$ partitions $V(G)$ into L sets, one can adapt our approach and study the LS$_+$-relaxations of G via analyzing L-dimensional shadows of these sets.

3.3. **Strategy for the proof of the main result.** So far, we have an infinite family of graphs $\{H_k\}$ with nice symmetries. In Lemma 8(i) we derived a family of facets of STAB(H_k). Then, in Proposition 12, we established a useful tool which guarantees for every integer $k \geq 2$, the existence of a symmetrized point x' in $\mathrm{LS}_+^p(H_k)$ using the existence of an arbitrary point x in $\mathrm{LS}_+^p(H_k)$, where p is an arbitrary non-negative integer. The dual counterpart of this symmetrization is given by Lemma 8(ii) which presents a (symmetrized) valid inequality for STAB(H_k). Due to the symmetries of H_k, these graphs admit $\mathcal{A}$-balancing automorphisms which provide us with points $x' \in \mathrm{LS}_+^p(H_k)$ and valid inequalities $a^\top x \leq \beta$ of STAB(H_k) of high LS$_+$-rank, such that x' is an optimal solution (or a nearly optimal solution) of

$$\max\left\{a^\top x \ : \ x \in \mathrm{LS}_+^p(H_k)\right\},$$

and x' as well as the normal vector a of high LS$_+$-rank valid inequality both have at most two distinct entries, each. Therefore, for every non-negative integer p and for every integer $k \geq 2$, we can focus on a two dimensional shadow $\Phi(\mathrm{LS}_+^p(H_k))$ of $\mathrm{LS}_+^p(H_k)$. We also have a complete characterization of $\Phi(\mathrm{STAB}(H_k))$ for every integer $k \geq 2$ (Lemma 15).

The rest of the proof of the main result, presented in the next section, starts by characterizing all symmetric positive semidefinite matrices that could certify the membership of a vector x' in $\mathrm{LS}_+(H_k)$ (x', as above, has at most two distinct entries). This characterization is relatively simple, since due to the isolated symmetries of x', there always exists a symmetric positive semidefinite certificate matrix for certifying the membership of x' in $\mathrm{LS}_+(H_k)$ which has at most four distinct entries (ignoring the 00^{th} entry of the certificate matrix, which is one). Next, we construct a compact convex set which is described by three linear inequalities and a quadratic inequality such that this set is a subset of $\Phi(\mathrm{LS}_+(H_k))$ and a strict superset of $\Phi(\mathrm{STAB}(H_k))$ for every integer $k \geq 4$. This enables us to conclude for all $k \geq 4$, $r_+(H_k) \geq 2$, and establish some of the tools for the rest of the proof. The point $w_k\left(\frac{1}{k}, \frac{k-1}{k}\right)$ already plays a special role.

Then, we put all these tools together to prove that there is a point in $\mathrm{LS}_+^p(H_k) \setminus \mathrm{STAB}(H_k)$ which is very close to $w_k\left(\frac{1}{k}, \frac{k-1}{k}\right)$ (we do this partly by working in the 2-dimensional space where $\Phi(\mathrm{LS}_+^p(H_k)$ lives). For the recursive construction of certificate matrices (to establish membership in $\mathrm{LS}_+^p(H_k)$), we show that in addition to the matrix being symmetric, positive semidefinite, and satisfying a simple linear inequality, membership of two suitable vectors $w_{k-1}(a_1, b_1)$ and $w_{k-1}(a_2, b_2)$ in $\mathrm{LS}_+^{p-1}(H_{k-1})$ suffice (Lemma 19). The rest of the analysis proves that there exist values for these parameters which allow the construction of certificate matrices for suitable pairs of integers k and p.

4. Proof of the main result

4.1. $\Phi(\mathrm{LS}_+(H_k))$ — **the shadow of the first relaxation.** Next, we aim to study the set $\Phi(\mathrm{LS}_+(H_k))$. To do that, we first look into potential certificate matrices for $w_k(a,b)$ that have plenty of symmetries. Given $k \in \mathbb{N}$ and $a,b,c,d \in \mathbb{R}$, we define the matrix $W_k(a,b,c,d) \in$

$\mathbb{R}^{(3k+1)\times(3k+1)}$ such that $W_k(a,b,c,d) := \begin{bmatrix} 1 & w_k(a,b)^\top \\ w_k(a,b) & \overline{W} \end{bmatrix}$, where

$$\overline{W} := \begin{bmatrix} a & 0 & a-c \\ 0 & b & 0 \\ a-c & 0 & a \end{bmatrix} \otimes I_k + \begin{bmatrix} c & a-c & 0 \\ a-c & d & a-c \\ 0 & a-c & c \end{bmatrix} \otimes (J_k - I_k).$$

Note that $\otimes$ denotes the Kronecker product, I_k is the k-by-k identity matrix, and J_k is the k-by-k matrix of all ones. Also, the columns of $\overline{W}$ are indexed by the vertices $1_0, 1_1, 1_2, 2_0, 2_1, 2_2, \ldots$ from left to right, with the rows following the same ordering. Then we have the following.

Lemma 16. *Let $k \in \mathbb{N}$ and $a,b,c,d \in \mathbb{R}$. Then $W_k(a,b,c,d) \succeq 0$ if and only if all of the following holds:*

(S1) $c \geq 0$;
(S2) $a - c \geq 0$;
(S3) $(b-d) - (a-c) \geq 0$;
(S4) $2a + (k-2)c - 2ka^2 \geq 0$;
(S5) $(2a + (k-2)c - 2ka^2)(2b + 2(k-1)d - 2kb^2) - (2(k-1)(a-c) - 2kab)^2 \geq 0$.

Proof. Define matrices $\overline{W}_1, \overline{W}_2, \overline{W}_3 \in \mathbb{R}^{3k \times 3k}$ where

$$\overline{W}_1 := \frac{1}{2} \cdot J_k \otimes \begin{bmatrix} c & 0 & -c \\ 0 & 0 & 0 \\ -c & 0 & c \end{bmatrix},$$

$$\overline{W}_2 := \frac{1}{k} \cdot (kI_k - J_k) \otimes \begin{bmatrix} a-c & c-a & a-c \\ c-a & b-d & c-a \\ a-c & c-a & a-c \end{bmatrix},$$

$$\overline{W}_3 := \frac{1}{2k} \cdot J_k \otimes \begin{bmatrix} 2a+(k-2)c-2ka^2 & 2(k-1)(a-c)-2kab & 2a+(k-2)c-2ka^2 \\ 2(k-1)(a-c)-2kab & 2b+2(k-1)d-2kb^2 & 2(k-1)(a-c)-2kab \\ 2a+(k-2)c-2ka^2 & 2(k-1)(a-c)-2kab & 2a+(k-2)c-2ka^2 \end{bmatrix}.$$

Then

$$\begin{aligned} &\overline{W}_1 + \overline{W}_2 + \overline{W}_3 \\ &= \begin{bmatrix} a-a^2 & -ab & a-c-a^2 \\ -ab & b-b^2 & -ab \\ a-c-a^2 & -ab & a-ab^2 \end{bmatrix} \otimes I_k + \begin{bmatrix} c-a^2 & a-c-ab & -a^2 \\ a-c-ab & d-b^2 & a-c-ab \\ -a^2 & a-c-ab & c-a^2 \end{bmatrix} \otimes (J_k - I_k) \\ &= \overline{W} - w_k(a,b)(w_k(a,b))^\top, \end{aligned}$$

which is a Schur complement of $W_k(a,b,c,d)$. Moreover, observe that the columns of $\overline{W}_i$ and $\overline{W}_j$ are orthogonal whenever $i \neq j$. Thus, we see that $W_k(a,b,c,d) \succeq 0$ if and only if $\overline{W}_1, \overline{W}_2$, and $\overline{W}_3$ are all positive semidefinite. Now observe that

$$\overline{W}_1 \succeq 0 \iff \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \succeq 0 \iff \text{(S1)},$$

$$\overline{W}_2 \succeq 0 \iff \begin{bmatrix} a-c & c-a \\ c-a & b-d \end{bmatrix} \succeq 0 \iff \text{(S2) and (S3)},$$

$$\overline{W}_3 \succeq 0 \iff \begin{bmatrix} 2a+(k-2)c-2ka^2 & 2(k-1)(a-c)-2kab \\ 2(k-1)(a-c)-2kab & 2b+2(k-1)d-2kb^2 \end{bmatrix} \succeq 0 \iff \text{(S4) and (S5)}.$$

Thus, the claim follows. $\square$

Next, for convenience, define $q_k := 1 - \sqrt{\frac{k}{2k-2}}$, and

$$p_k(x,y) := (2x^2 - x) + 2q_k^2(y^2 - y) + 4q_k xy.$$

Notice that the curve $p_k(x,y) = 0$ is a parabola for all $k \geq 3$. Then, using Lemma 16, we have the following.

Proposition 17. *For every $k \geq 4$,*

$$\Phi(\mathrm{LS}_+(H_k)) \supseteq \left\{(x,y) \in \mathbb{R}^2 : p_k(x,y) \leq 0, x+y \leq 1, x \geq 0, y \geq 0\right\}. \tag{9}$$

Proof. For convenience, let C denote the set on the right hand side of (9). Notice that the boundary points of the triangle $\{(x,y) : x+y \leq 1, x \geq 0, y \geq 0\}$ which lie in C are also boundary points of $\Phi(\mathrm{STAB}(H_k))$. Thus, let us define the set of points

$$C_0 := \left\{(x,y) \in \mathbb{R}^2 : p_k(x,y) = 0, \frac{1}{k} < x < \frac{1}{2}\right\}.$$

To prove our claim, it suffices to prove that for all $(a,b) \in C_0$, there exist $c,d \in \mathbb{R}$ such that $W_k(a,b,c,d)$ certifies $w_k(a,b) \in \mathrm{LS}_+(H_k)$.

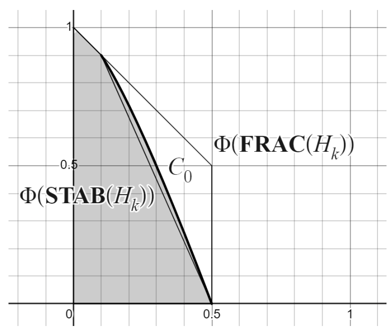

FIGURE 3. Visualizing the set C for the case $k = 10$

To help visualize our argument, Figure 3 (which is produced using Desmos' online graphing calculator [Des]) illustrates the set C for the case of $k = 10$.

Now given $(a,b) \in \mathbb{R}^2$ (not necessarily in C_0), consider the conditions (S3) and (S5) from Lemma 16:

$$b - a + c \geq d, \tag{10}$$

$$(2a + (k-2)c - 2ka^2)(2b + 2(k-1)d - 2kb^2) - (2(k-1)(a-c) - 2kab)^2 \geq 0. \tag{11}$$

If we substitute $d = b - a + c$ into (11) and solve for c that would make both sides equal, we would obtain the quadratic equation $p_2c^2 + p_1c + p_0 = 0$ where

$$\begin{aligned}
p_2 &:= (k-2)(2(k-1)) - (-2(k-1))^2,\\
p_1 &:= (k-2)(2b + 2(k-1)(b-a) - 2kb^2) + (2a - 2ka^2)(2(k-1))\\
&\quad - 2(-2(k-1))(2(k-1)a - 2kab),\\
p_0 &:= (2a - 2ka^2)(2b + 2(k-1)(b-a) - 2kb^2) - (2(k-1)a - 2kab)^2.
\end{aligned}$$

We then define

$$c := \frac{-p_1}{2p_2} = -a^2 - 2ab - \frac{b^2}{2} + \frac{3a}{2} + \frac{b}{2} + \frac{b\,(b-1)}{2\,(k-1)},$$

and $d := b - a + c$. We claim that, for all $(a,b) \in C_0$, $W_k(a,b,c,d)$ would certify $w_k(a,b) \in \mathrm{LS}_+(H_k)$. First, we provide some intuition for the choice of c. Let $\overline{q}_k := 1 + \sqrt{\frac{k}{2k-2}}$ and

$$\overline{p}_k(x,y) := (2x^2 - x) + 2\overline{q}_k^2(y^2 - y) + 4\overline{q}_k xy.$$

Then, if we consider the discriminant $\Delta p := p_1^2 - 4p_0p_2$, one can check that

$$\Delta p = 4(k-1)^2 p_k(a,b)\overline{p}_k(a,b).$$

Thus, when $\Delta p > 0$, there would be two solutions to the quadratic equation $p_2x^2 + p_1x + p_0 = 0$, and c would be defined as the midpoint of these solutions. In particular, when $(a,b) \in C_0$, $p_k(a,b) = \Delta p = 0$, and so $c = \frac{-p_1}{2p_2}$ would indeed be the unique solution that satisfies both (10) and (11) with equality.

Now we verify that $Y := W_k(a,b,c,d)$, as defined, satisfies all the conditions imposed by LS_+. We first show that $W_k(a,b,c,d) \succeq 0$ by verifying the conditions from Lemma 16. Notice that (S3) and (S5) must hold by the choice of c and d. Next, we check (S1), namely $c \geq 0$. Define the region

$$T := \left\{ (x,y) \in \mathbb{R}^2 : \frac{1}{k} \leq x \leq \frac{1}{2}, \frac{(1-2x)(k-1)}{k-2} \leq y \leq 1 - x \right\}.$$

In other words, T is the triangular region with vertices $\left(\frac{1}{k}, \frac{k-1}{k}\right), \left(\frac{1}{2}, 0\right)$, and $\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus, T contains C_0. and it suffices to show that $c \geq 0$ over T. Fixing k and viewing c as a function of a and b, we obtain

$$\frac{\partial c}{\partial a} = -2a - 2b + \frac{3}{2}, \quad \frac{\partial c}{\partial b} = \frac{(-4a - 2b + 1)k + 4a + 4b - 2}{2k - 2}.$$

Solving $\frac{\partial c}{\partial a} = \frac{\partial c}{\partial b} = 0$, we obtain the unique solution $(a,b) = \left(\frac{-k+2}{4k}, \frac{4k-2}{4k}\right)$, which is outside of T. Next, one can check that c is non-negative over the three edges of T, and we conclude that $c \geq 0$ over T, and thus (S1) holds. The same approach also shows that both $a - c$ and $2a + (k-2)c - 2ka^2$ are non-negative over T, and thus (S2) and (S4) hold as well, and we conclude that $Y \succeq 0$.

Next, we verify that $Ye_i, Y(e_0 - e_i) \in \mathrm{cone}(\mathrm{FRAC}(H_k))$. By the symmetry of H_k, it suffices to verify these conditions for the vertices $i = 1_0$ and $i = 1_1$.

- Ye_{1_0}: Define $S_1 := \{1_0, 1_2\} \cup \{i_1 : 2 \leq i \leq k\}$ and $S_2 := [k]_0$. Observe that both S_1, S_2 are stable sets of H_k, and that

$$Ye_{1_0} = (a-c)\begin{bmatrix} 1 \\ \chi_{S_1} \end{bmatrix} + c\begin{bmatrix} 1 \\ \chi_{S_2} \end{bmatrix}, \tag{12}$$

  Since we verified above that $c \geq 0$ and $a - c \geq 0$, $Ye_{1_0} \in \mathrm{cone}(\mathrm{STAB}(H_k))$, which is contained in $\mathrm{cone}(\mathrm{FRAC}(H_k))$.
- Ye_{1_1}: The non-trivial edge inequalities imposed by $Ye_{1_1} \in \mathrm{cone}(\mathrm{FRAC}(H_k))$ are

$$[Ye_{1_1}]_{2_2} + Y[e_{1_1}]_{3_0} \leq [Ye_{1_1}]_0 \Rightarrow 2(a-c) \leq b, \tag{13}$$

$$[Ye_{1_1}]_{2_0} + [Ye_{1_1}]_{2_1} \leq [Ye_{1_1}]_0 \Rightarrow a - c + d \leq b. \tag{14}$$

  Note that (14) is identical to (S3), which we have already established. Next, we know from (S4) that $c \geq \frac{2ka^2 - 2a}{k-2}$. That together with the fact that $2(k-1)a + (k-2)b \geq k - 1$ for all $(a,b) \in C_0$ and $k \geq 4$ implies (13).

- $Y(e_0 - e_{1_0})$: The non-trivial edge inequalities imposed by $Y(e_0 - e_{1_0}) \in \operatorname{cone}(\mathrm{FRAC}(H_k))$ are

$$[Y(e_0 - e_{1_0})]_{2_2} + [Y(e_0 - e_{1_0})]_{3_0} \leq [Y(e_0 - e_{1_0})]_0 \Rightarrow a + (a - c) \leq 1 - a, \tag{15}$$

$$[Y(e_0 - e_{1_0})]_{2_1} + [Y(e_0 - e_{1_0})]_{2_2} \leq [Y(e_0 - e_{1_0})]_0 \Rightarrow (b - a + c) + a \leq 1 - a. \tag{16}$$

(15) follows from (13) and the fact that $a - c \geq 0$. For (16), we aim to show that $a + b + c \leq 1$. Define the quantity

$$g(x, y) := 1 - \frac{5}{2}x - \frac{3}{2}y + x^2 + \frac{1}{2}y^2 + 2xy - \frac{y\,(y-1)}{2\,(k-1)}.$$

Then $g(a, b) = 1 - a - b - c$. Notice that, for all k, the curve $g(x, y) = 0$ intersects with C at exactly three points: $(0, 1), \left(\frac{1}{k}, \frac{k-1}{k}\right)$, and $\left(\frac{1}{2}, 0\right)$. In particular, the curve does not intersect the interior of C. Therefore, $g(x, y)$ is either non-negative or non-positive over C. Since $g(0, 0) = 1$, it is the former. Hence, $g(x, y) \geq 0$ over C (and hence C_0), and (16) holds.

- $Y(e_0 - e_{1_1})$: The non-trivial edge inequalities imposed by $Y(e_0 - e_{1_1}) \in \operatorname{cone}(\mathrm{FRAC}(H_k))$ are

$$[Y(e_0 - e_{1_1})]_{1_0} + [Y(e_0 - e_{1_1})]_{2_2} \leq [Y(e_0 - e_{1_1})]_0 \Rightarrow a + c \leq 1 - b, \tag{17}$$

$$[Y(e_0 - e_{1_1})]_{2_0} + [Y(e_0 - e_{1_1})]_{2_1} \leq [Y(e_0 - e_{1_1})]_0 \Rightarrow c + (b - d) \leq 1 - b. \tag{18}$$

(17) is identical to (15), which we have verified above. Finally, (18) follows from (S3) and the fact that $a + b \leq 1$ for all $(a, b) \in C$.

This completes the proof. □

An immediate consequence of Proposition 17 is the following.

Corollary 18. *For all $k \geq 4$, $r_+(H_k) \geq 2$.*

Proof. For every $k \geq 4$, the set described in (9) is not equal to $\Phi(\mathrm{STAB}(H_k))$. Thus, there exists $w_k(a, b) \in \mathrm{LS}_+(H_k) \setminus \mathrm{STAB}(H_k)$ for all $k \geq 4$, and the claim follows. □

Corollary 18 is sharp — notice that destroying any vertex in H_3 yields a bipartite graph, so it follows from Theorem 4(i) that $r_+(H_3) = 1$. Also, since destroying a vertex in H_4 either results in H_3 or a bipartite graph, we see that $r_+(H_4) = 2$.

4.2. **Showing $r_+(H_k) = \Theta(k)$.** We now develop a few more tools that we need to establish the main result of this section. Again, to conclude that $r_+(H_k) > p$, it suffices to show that $\Phi(\mathrm{LS}_+^p(H_k)) \supset \Phi(\mathrm{STAB}(H_k))$. In particular, we will do so by finding a point in $\Phi(\mathrm{LS}_+^p(H_k)) \setminus \Phi(\mathrm{STAB}(H_k))$ that is very close to the point $\left(\frac{1}{k}, \frac{k-1}{k}\right)$. Given $(a, b) \in \mathbb{R}^2$, let

$$s_k(a, b) := \frac{\frac{k-1}{k} - b}{\frac{1}{k} - a}.$$

That is, $s_k(a, b)$ is the slope of the line that contains the points (a, b) and $\left(\frac{1}{k}, \frac{k-1}{k}\right)$. Next, define

$$f(k, p) := \sup\left\{ s_k(a, b) : (a, b) \in \Phi(\mathrm{LS}_+^p(H_k)), a > \frac{1}{k} \right\}.$$

In other words, $f(k, p)$ is the slope of the tangent line to $\Phi(\mathrm{LS}_+^p(H_k))$ at the point $\left(\frac{1}{k}, \frac{k-1}{k}\right)$ towards the right hand side. Thus, for all $\ell < f(k, p)$, there exists $\epsilon > 0$ where the point $\left(\frac{1}{k} + \epsilon, \frac{k-1}{k} + \ell\epsilon\right)$ belongs to $\Phi(\mathrm{LS}_+^p(H_k))$. For $p = 0$ (and so $\mathrm{LS}_+^p(H_k) = \mathrm{FRAC}(H_k)$), observe that $f(k, 0) = -1$ for all $k \geq 2$ (attained by the point $\left(\frac{1}{2}, \frac{1}{2}\right)$). Next, for $p = 1$, consider the polynomial $p_k(x,y)$ defined before Proposition 17. Then any point (x,y) on the curve $p_k(x,y)=0$ has slope

$$\frac{\partial}{\partial x} p_k(x,y) = \frac{1-4x-4q_k y}{4q_k^2 y - q_k + 4q_k x}.$$

Thus, by Proposition 17,

$$f(k,1) \geq \left. \frac{\partial}{\partial x} p_k(x,y) \right|_{(x,y)=\left(\frac{1}{k}, \frac{k-1}{k}\right)} = -1 - \frac{k}{3k^2 - 2(k-1)^2\sqrt{\frac{2k}{k-1}} - 4k} \tag{19}$$

for all $k \geq 4$. Finally, if $p \geq r_+(H_k)$, then $f(k,p) = -\frac{2(k-1)}{k-2}$ (attained by the point $\left(\frac{1}{2}, 0\right) \in \Phi(\text{STAB}(H_k))$).

We will prove our LS_+-rank lower bound on H_k by showing that $f(k,p) > -\frac{2(k-1)}{k-2}$ for some $p = \Theta(k)$. To do so, we first show that the recursive structure of H_k allows us to establish $(a,b) \in \Phi(\text{LS}_+^p(H_k))$ by verifying (among other conditions) the membership of two particular points in $\Phi(\text{LS}_+^{p-1}(H_{k-1}))$, which will help us relate the quantities $f(k-1,p-1)$ and $f(k,p)$. Next, we bound the difference $f(k-1,p-1) - f(k,p)$ from above, which implies that it takes LS_+ many iterations to knock the slopes $f(k,p)$ from that of $\Phi(\text{FRAC}(H_k))$ down to that of $\Phi(\text{STAB}(H_k))$.

First, here is a tool that will help us verify certificate matrices recursively.

Lemma 19. *Suppose $a,b,c,d \in \mathbb{R}$ satisfy all of the following:*

(i) $W_k(a,b,c,d) \succeq 0$,
(ii) $2b + 2c - d \leq 1$,
(iii) $w_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right), w_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right) \in \text{LS}_+^{p-1}(H_{k-1})$.

Then $W_k(a,b,c,d)$ certifies $w_k(a,b) \in \text{LS}_+^p(H_k)$.

Proof. For convenience, let $Y := W_k(a,b,c,d)$. First, $Y \succeq 0$ from (i). Next, we focus on the following column vectors:

- Ye_{1_0}: $Y \succeq 0$ implies that $c \geq 0$ and $a - c \geq 0$ by Lemma 16. Then it follows from (12) that $Ye_{1_0} \in \text{cone}(\text{STAB}(H_k)) \subseteq \text{cone}(\text{LS}_+^{p-1}(H_k))$.
- Ye_{1_1}: (iii) implies $\begin{bmatrix} b \\ w_{k-1}(a-c,d) \end{bmatrix} \in \text{cone}(\text{LS}_+^{p-1}(H_{k-1}))$. Thus,

$$Ye_{1_1} = \begin{bmatrix} b \\ 0 \\ b \\ 0 \\ w_{k-1}(a-c,d) \end{bmatrix} \in \text{cone}(\text{LS}_+^{p-1}(H_k)).$$

- $Y(e_0 - e_{1_0})$: Let $S_1 := [k]_2$, which is a stable set in H_k. Then observe that

$$Y(e_0 - e_{1_0}) = c \begin{bmatrix} 1 \\ \chi_{S_1} \end{bmatrix} + \begin{bmatrix} 1-a-c \\ 0 \\ b \\ 0 \\ w_{k-1}(a-c, b-a+c) \end{bmatrix}.$$

By (iii) and the fact that $\text{cone}(\text{LS}_+^{p-1}(H_k))$ is a convex cone, it follows that $Y(e_0 - e_{1_0}) \in \text{cone}(\text{LS}_+^{p-1}(H_k))$.

- $Y(e_0 - e_{1_1})$: Define $S_2 := [k]_0, S_3 := \{1_0, 1_2\} \cup \{i_1 : 2 \leq i \leq k\}$, and $S_4 := \{i_1 : 2 \leq i \leq k\}$, which are all stable sets in H_k. Now observe that
$$Y(e_0 - e_{1_1}) = c \begin{bmatrix} 1 \\ \chi_{S_1} \end{bmatrix} + c \begin{bmatrix} 1 \\ \chi_{S_2} \end{bmatrix} + (a-c) \begin{bmatrix} 1 \\ \chi_{S_3} \end{bmatrix} + (b-d-a+c) \begin{bmatrix} 1 \\ \chi_{S_4} \end{bmatrix} + (1-2b-2c+d) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
Since $Y \succeq 0$, $b-d-a+c \geq 0$ from (S3). Also, $1-2b-2c+d \geq 0$ by (ii). Thus, $Y(e_0-e_{1_1})$ is a sum of vectors in cone(STAB(H_k)), and thus belongs to cone($\mathrm{LS}_+^{p-1}(H_k)$).

By the symmetry of H_k and $W_k(a,b,c,d)$, it suffices to verify the membership conditions for the above columns. Thus, it follows that $W_k(a,b,c,d)$ indeed certifies $w_k(a,b) \in \mathrm{LS}_+^p(H_k)$. □

Example 20. We illustrate Lemma 19 by using it to show that $r_+(H_7) \geq 3$. Let $k = 7, a = 0.1553, b = 0.8278, c = 0.005428$, and $d = 0.6665$. Then one can check (via Lemma 16) that $W_k(a,b,c,d) \succeq 0$, and $2b + 2c - d \leq 1$. Also, one can check that $w_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right)$ and $w_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right)$ both belong to $\mathrm{LS}_+(H_{k-1})$ using Proposition 17. Thus, Lemma 19 applies, and $w_k(a,b) \in \mathrm{LS}_+^2(H_k)$. Now observe that $2(k-1)a + (k-2)b = 6.0026 > k-1$, and so $w_k(a,b) \notin \mathrm{STAB}(H_k)$, and we conclude that $r_+(H_7) \geq 3$.

Next, we apply Lemma 19 iteratively to find a lower bound for the LS$_+$-rank of H_k as a function of k. The following is an updated version of Lemma 19 that gets us a step closer to directly relating $f(k,p)$ and $f(k-1,p-1)$.

Lemma 21. *Suppose $a,b,c,d \in \mathbb{R}$ satisfy all of the following:*

(i) $W_k(a,b,c,d) \succeq 0$,
(ii) $2b + 2c - d \leq 1$,
(iii) $\max\left\{ s_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right), s_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right)\right\} \leq f(k-1, p-1)$.

Then $f(k,p) \geq s_k(a,b)$.

Proof. Given $a,b,c,d \in \mathbb{R}$ that satisfy the given assumptions, define
$$a(\lambda) := \frac{\lambda}{k} + (1-\lambda)a, \qquad b(\lambda) := \frac{\lambda(k-1)}{k} + (1-\lambda)b,$$
$$c(\lambda) := (1-\lambda)c, \qquad d(\lambda) := \frac{\lambda(k-2)}{k} + (1-\lambda)d.$$
Then notice that
$$W_k(a(\lambda), b(\lambda), c(\lambda), d(\lambda)) = \lambda W_k\left(\frac{1}{k}, \frac{k-1}{k}, 0, \frac{k-2}{k}\right) + (1-\lambda) W_k(a,b,c,d). \tag{20}$$
Once can check (e.g., via Lemma 16) that $W_k\left(\frac{1}{k}, \frac{k-1}{k}, 0, \frac{k-2}{k}\right) \succeq 0$ for all $k \geq 2$. Since $W_k(a,b,c,d) \succeq 0$ from (i), it follows from (20) and the convexity of the positive semidefinite cone that
$W_k(a(\lambda), b(\lambda), c(\lambda), d(\lambda)) \succeq 0$ for all $\lambda \in [0,1]$.

Now observe that for all $\lambda > 0$, $s_k(a(\lambda), b(\lambda)) = s_k(a,b)$, $s_{k-1}\left(\frac{a(\lambda)-c(\lambda)}{b(\lambda)}, \frac{d(\lambda)}{b(\lambda)}\right) = s_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right)$, and $s_{k-1}\left(\frac{a(\lambda)-c(\lambda)}{1-a(\lambda)-c(\lambda)}, \frac{b(\lambda)-a(\lambda)+c(\lambda)}{1-a(\lambda)-c(\lambda)}\right) = s_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right)$. By assumption (iii), there must be a sufficiently small $\lambda > 0$ where $w_{k-1}\left(\frac{a(\lambda)-c(\lambda)}{b(\lambda)}, \frac{d(\lambda)}{b(\lambda)}\right)$ and $w_{k-1}\left(\frac{a(\lambda)-c(\lambda)}{1-a(\lambda)-c(\lambda)}, \frac{b(\lambda)-a(\lambda)+c(\lambda)}{1-a(\lambda)-c(\lambda)}\right)$ are both contained in $\mathrm{LS}_+^{p-1}(H_k)$. Then Lemma 19 implies that $w_k(a(\lambda), b(\lambda)) \in \mathrm{LS}_+^p(H_k)$, and the claim follows. □

Next, we define four values corresponding to each k that will be important in our subsequent argument:

$$u_1(k) := -\frac{2(k-1)}{k-2}, \qquad u_2(k) := \frac{k-4-\sqrt{17k^2-48k+32}}{2(k-2)},$$

$$u_3(k) := \frac{4(k-1)(-3k+4-2\sqrt{k-1})}{(k-2)(9k-10)}, \qquad u_4(k) := -1 - \frac{k}{3k^2-2(k-1)^2\sqrt{\frac{2k}{k-1}}-4k}.$$

Notice that $u_1(k) = f(k,p)$ for all $p \geq r_+(H_k)$, and $u_4(k)$ is the expression given in (19), the lower bound for $f(k,1)$ that follows from Proposition 17. Then we have the following.

Lemma 22. *For every $k \geq 5$,*

$$u_1(k) < u_2(k) < u_3(k) < u_4(k).$$

Proof. First, one can check that the chain of inequalities holds when $5 \leq k \leq 26$, and that

$$-2 < u_2(k) < \frac{1-\sqrt{17}}{2} < u_3(k) < -\frac{4}{3} < u_4(k) \tag{21}$$

holds for $k = 27$. Next, notice that

$$\lim_{k\to\infty} u_1(k) = -2, \quad \lim_{k\to\infty} u_2(k) = \frac{1-\sqrt{17}}{2}, \quad \lim_{k\to\infty} u_3(k) = -\frac{4}{3},$$

and that $u_i(k)$ is an increasing function of k for all $i \in [4]$. Thus, (21) in fact holds for all $k \geq 27$, and our claim follows. □

Now we are ready to prove the following key lemma which bounds the difference between $f(k-1,p-1)$ and $f(k,p)$.

Lemma 23. *Given $k \geq 5$ and $\ell \in (u_1(k), u_3(k))$, let*

$$\gamma := (k-2)(9k-10)\ell^2 + 8(k-1)(3k-4)\ell + 16(k-1)^2,$$

and

$$h(k,\ell) := \frac{4(k-2)\ell + 8(k-1)}{\sqrt{\gamma} + 3(k-2)\ell + 8(k-1)} - 2 - \ell.$$

If $f(k-1,p-1) \leq \ell + h(k,\ell)$, then $f(k,p) \leq \ell$.

Proof. Given $\epsilon > 0$, define $a := \frac{1}{k} + \epsilon$ and $b := \frac{k-1}{k} + \ell\epsilon$. We solve for c, d so that they satisfy condition (ii) in Lemma 21 and (S5) in Lemma 16 with equality. That is,

$$d - 2b - 2c = 1, \tag{22}$$

$$(2a + (k-2)c - 2ka^2)(2b + 2(k-1)d - 2kb^2) - (2(k-1)(a-c) - 2kab)^2 = 0. \tag{23}$$

To do so, we substitute $d = 2b + 2c - 1$ into (23), and obtain the quadratic equation

$$p_2c^2 + p_1c + p_0 = 0$$

where

$$\begin{aligned}
p_2 &:= (k-2)(4(k-1)) - (-2(k-1))^2, \\
p_1 &:= (k-2)(2b + 2(k-1)(2b-1) - 2kb^2) + (2a - 2ka^2)(4(k-1)) \\
&\qquad - 2(-2(k-1))(2(k-1)a - 2kab), \\
p_0 &:= (2a - 2ka^2)(2b + 2(k-1)(2b-1) - 2kb^2) - (2(k-1)a - 2kab)^2.
\end{aligned}$$

We then define $c := \frac{-p_1+\sqrt{p_1^2-4p_0p_2}}{2p_2}$ (this would be the smaller of the two solutions, as $p_2 < 0$), and $d := 2b+2c-1$. First, we assure that c is well defined. If we consider the discriminant $\Delta p := p_1^2 - 4p_0p_2$ as a function of ϵ, then $\Delta p(0) = 0$, and that $\frac{d^2}{d\epsilon^2}\Delta p(0) > 0$ for all $\ell \in (u_1(k), u_3(k))$. Thus, there must exist $\epsilon > 0$ where $\Delta p \geq 0$, and so c, d are well defined.

Next, we verify that $W_k(a, b, c, d) \succeq 0$ for some $\epsilon > 0$ by checking the conditions from Lemma 16. First, by the choice of c, d, (S5) must hold. Next, define the quantities

$$\theta_1 := c, \qquad \theta_2 := a - c,$$
$$\theta_3 := b - d - a + c, \qquad \theta_4 := 2a + (k-2)c - 2ka^2.$$

Notice that at $\epsilon = 0$, $\theta_i = 0$ for all $i \in [4]$. Next, given a quantity q that depends on ϵ, we use the notation $q'(0)$ denote the one-sided derivative $\lim_{\epsilon \to 0^+} \frac{q}{\epsilon}$. Then it suffices to show that $\theta_i'(0) \geq 0$ for all $i \in [4]$. Observe that

$$\theta_1'(0) \geq 0 \iff c'(0) \geq 0, \qquad \theta_2'(0) \geq 0 \iff c'(0) \leq 1,$$
$$\theta_3'(0) \geq 0 \iff c'(0) \leq -1-\ell, \qquad \theta_4'(0) \geq 0 \iff c'(0) \geq \frac{2}{k-2}.$$

Now one can check that

$$c'(0) = \frac{-3k\ell - \sqrt{\gamma} - 4k + 2\ell + 4}{4k-4}.$$

As a function of ℓ, $c'(0)$ is increasing over $(u_1(k), u_3(k))$, with

$$c'(0)\big|_{\ell = u_1(k)} = \frac{2}{k-2}, \qquad c'(0)\big|_{\ell = u_3(k)} = \frac{(6k-4)\sqrt{k-1} + 10k - 12}{(k-2)(9k-10)}.$$

Thus, for all $k \geq 5$, we see that $\frac{2}{k-2} \leq c'(0) \leq \min\{1, -1-\ell\}$ for all $\ell \in (u_1(k), u_3(k))$, and so there exists $\epsilon > 0$ where $W_k(a, b, c, d) \succeq 0$.

Next, for convenience, let

$$s_1 := s_{k-1}\left(\frac{a-c}{b}, \frac{d}{b}\right), \qquad s_2 := s_{k-1}\left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right).$$

Notice that both s_1, s_2 are undefined at $\epsilon = 0$, as $\left(\frac{a-c}{b}, \frac{d}{b}\right) = \left(\frac{a-c}{1-a-c}, \frac{b-a+c}{1-a-c}\right) = \left(\frac{1}{k-1}, \frac{k-2}{k-1}\right)$ in this case. Now one can check that

$$\lim_{\epsilon \to 0^+} s_1 = \frac{-2\sqrt{\gamma} - 2(k-2)\ell - 8(k-1)}{\sqrt{\gamma} + 3(k-2)\ell + 8(k-1)},$$
$$\lim_{\epsilon \to 0^+} s_2 = \frac{(-2k+3)\sqrt{\gamma} - (2k-1)(k-2)\ell - 8(k-1)^2}{(k-2)\sqrt{\gamma} + (3k-2)(k-2)\ell + 8(k-1)^2}.$$

Observe that for $k \geq 5$ and for all $\ell \in (u_1(k), 0)$, we have

$$0 > \frac{-2\sqrt{\gamma}}{\sqrt{\gamma}} > \frac{(-2k+3)\sqrt{\gamma}}{(k-2)\sqrt{\gamma}},$$
$$0 > \frac{-2(k-2)\ell - 8(k-1)}{3(k-2)\ell + 8(k-1)} > \frac{-(2k-1)(k-2)\ell - 8(k-1)^2}{(3k-2)(k-2)\ell + 8(k-1)^2}.$$

Thus, we conclude that for all k, ℓ under our consideration, $s_1 \geq s_2$ for arbitrarily small $\epsilon > 0$.

Now, notice that $h(k, \ell) = \lim_{\epsilon \to 0^+} s_1 - \ell$. Thus, if $\ell \in (u_1(k), u_3(k))$, then there exists $\epsilon > 0$ where the matrix $W_k(a, b, c, d)$ as constructed is positive semidefinite, satisfies $d \geq 2b + 2c - 1$ (by the choice of c, d), with $s_2 \leq s_1 \leq h(k, \ell) + \ell$. Hence, if $f(k-1, p-1) \geq \ell + h(k, \ell)$, then Lemma 21 applies, and we obtain that $f(k, p) \geq \ell$. $\square$

Applying Lemma 23 iteratively, we obtain the following.

Lemma 24. *Given $k \geq 5$, suppose there exists $\ell_1, \ldots, \ell_p \in \mathbb{R}$ where*

(i) *$\ell_p > u_1(k)$, $\ell_2 < u_3(k-p+2)$, and $\ell_1 < u_4(k-p+1)$;*
(ii) *$\ell_i + h(k-p+i, \ell_i) \leq \ell_{i-1}$ for all $i \in \{2, \ldots, p\}$.*

Then $r_+(H_k) \geq p+1$.

Proof. First, notice that $\ell_1 < u_4(k-p+1) \leq f(k-p+1, 1)$ by Proposition 17. Then since $\ell_2 < u_3(k-p+2)$ and $\ell_2 + h(k-p+2, \ell_2) \leq \ell_1$, Lemma 23 implies that $\ell_2 \leq f(k-p+2, 2)$. Iterating this argument results in $\ell_i \leq f(k-p+i, i)$ for every $i \in [p]$. In particular, we have $\ell_p \leq f(k, p)$. Since $\ell_p > u_1(k)$, it follows that $r_+(H_k) > p$, and the claim follows. □

Lemmas 23 and 24 provide a simple procedure of establishing LS_+-rank lower bounds for H_k.

Example 25. Let $k = 7$. Then $\ell_2 = -2.39$ and $\ell_1 = \ell_2 + h(7, \ell_2)$ certify that $r_+(H_7) \geq 3$. Similarly, for $k = 10$, one can let $\ell_3 = -2.24, \ell_2 = \ell_3 + h(10, \ell_3)$, and $\ell_1 = \ell_2 + h(9, \ell_2)$ and use Lemma 24 to verify that $r_+(H_{10}) \geq 4$.

Next, we prove a lemma that will help us obtain a lower bound for $r_+(H_k)$ analytically.

Lemma 26. *For all $k \geq 5$ and $\ell \in (u_1(k), u_2(k))$, $h(k, \ell) \leq \frac{2}{k-2}$.*

Proof. One can check that the equation $h(k, \ell) = \frac{2}{k-2}$ has three solutions: $\ell = u_1(k), u_2(k)$, and $\frac{k-4-\sqrt{17k^2-48k+32}}{2(k-2)}$ (which is greater than $u_2(k)$). Also, notice that $\frac{\partial}{\partial \ell} h(k, \ell)\big|_{\ell = u_1(k)} = -\frac{1}{k-1} < 0$. Since $h(k, \ell)$ is a continuous function of ℓ over $(u_1(k), u_2(k))$, it follows that $h(k, \ell) \leq \frac{2}{k-2}$ for all ℓ in this range. □

We are finally ready to prove the main result of this section.

Theorem 27. *The LS_+-rank of H_k is*

- *at least 2 for $4 \leq k \leq 6$;*
- *at least 3 for $7 \leq k \leq 9$;*
- *at least $\lfloor 0.19(k-2) \rfloor + 3$ for all $k \geq 10$.*

Proof. First, $r_+(H_4) \geq 2$ follows from Corollary 18, and $r_+(H_7) \geq 3$ was shown in Example 20 and again in Example 25. Moreover, one can use the approach illustrated in Example 25 to verify that $r_+(H_k) \geq \lfloor 0.19(k-2) \rfloor + 3$ for all k where $10 \leq k \leq 49$. Thus, we shall assume that $k \geq 50$ for the remainder of the proof.

Let $q \coloneqq \lfloor 0.19(k-2) \rfloor$, let $\epsilon > 0$ that we set to be sufficiently small, and define

$$\ell_i \coloneqq \epsilon + u_1(k) + \sum_{j=1}^{q+2-i} \frac{2}{k-1-j}.$$

for every $i \in [q+2]$. (We aim to subsequently apply Lemma 24 with $p = q+2$.) Now notice that

$$\sum_{j=1}^{q} \frac{2}{k-1-i} \leq \int_{k-2-q}^{k-2} \frac{2}{t} \, dt = 2 \ln \left(\frac{k-2}{k-2-q} \right),$$

Also, notice that

$$u_2(k-q) - u_1(k) \geq u_2\left(\frac{4}{5}k\right) - u_1(k),$$

as u_2 is an increasing function in k and $q \leq \frac{k}{5}$. Also, one can check that $\overline{w}(k) \coloneqq u_2\left(\frac{4}{5}k\right) - u_1(k)$ is also an increasing function for all $k \geq 5$. Next, we see that

$$2\ln\left(\frac{k-2}{k-2-q}\right) \leq \overline{w}(50) \iff q \leq \left(1 - \frac{1}{\exp(\overline{w}(50)/2)}\right)(k-2)$$

Since $1 - \frac{1}{\exp(\overline{w}(50)/2)} > 0.19$, the first inequality does hold by the choice of q. Hence,

$$\ell_2 - \epsilon = u_1(k) + \sum_{j=1}^{q} \frac{2}{k-1-j} < u_2(k-q).$$

Thus, we can choose ϵ sufficiently small so that $\ell_2 < u_2(k-q)$. Then Lemma 26 implies that $\ell_i + h(k-q-2+i, \ell_i) \leq \ell_{i-1}$ for all $i \in \{2, \ldots, q+2\}$. Also, for all $k \geq 50$, $u_2(k-q) + \frac{1}{k-q-1} < u_4(k-q-1)$. Thus, we obtain that $\ell_1 < u_4(k-q-1)$, and it follows from Lemma 24 that $r_+(H_k) \geq q+3$. □

Since H_k has $3k$ vertices, Theorem 27 (and the fact that $r_+(H_3) = 1$) readily implies Theorem 2. In other words, we now know that for every $\ell \in \mathbb{N}$, there exists a graph on no more than 16ℓ vertices that has LS$_+$-rank ℓ.

5. Chvátal–Gomory rank of STAB(H_k)

In this section we determine the degree of hardness of STAB(H_k) relative to another well-studied cutting plane procedure that is due to Chvátal [Chv73] with earlier ideas from Gomory [Gom58]. Given a set $P \subseteq [0,1]^n$, if $a^\top x \leq \beta$ is a valid inequality of P and $a \in \mathbb{Z}^n$, we say that $a^\top x \leq \lfloor \beta \rfloor$ is a *Chvátal–Gomory cut* for P. Then we define CG(P), the *Chvátal–Gomory closure* of P, to be the set of points that satisfy all Chvátal–Gomory cuts for P. Note that CG(P) is a closed convex set which contains all integral points in P. Furthermore, given an integer $p \geq 2$, we can recursively define $\mathrm{CG}^p(P) \coloneqq \mathrm{CG}(\mathrm{CG}^{p-1}(P))$. Then given any valid linear inequality of P_I, we can define its CG-*rank* (relative to P) to be the smallest integer p for which the linear inequality is valid for $\mathrm{CG}^p(P)$.

In Section 4, we proved that the inequality (4) has LS$_+$-rank $\Theta(|V(H_k)|)$. This implies that the inequality (3) also has LS$_+$-rank $\Theta(|V(H_k)|)$ (since it was shown in the proof of Lemma 8(ii) that (4) is a non-negative linear combination of (3)). Here, we show that (3) has CG-rank $\Theta(\log(|V(H_k)|))$.

Theorem 28. *Let d be the* CG*-rank of the facet* (3) *of* STAB(H_k) *relative to* FRAC(H_k)*. Then*

$$\log_4\left(\frac{3k-7}{2}\right) < d \leq \log_2(k-1).$$

Before providing a proof of Theorem 28, first, we need a lemma about the valid inequalities of STAB(H_k).

Lemma 29. *Suppose $a^\top x \leq \beta$ is valid for* STAB(H_k) *where $a \in \mathbb{Z}_+^{V(H_k)} \setminus \{0\}$. Then $\frac{\beta}{a^\top \bar{e}} > \frac{1}{3}$.*

Proof. We consider two cases. First, suppose that $a_{j_1} = 0$ for all $j \in [k]$. Since $[k]_p$ is a stable set in H_k for $p \in \{0,1,2\}$, observe that

$$a^\top \bar{e} = a^\top \left(\chi_{[k]_0} + \chi_{[k]_1} + \chi_{[k]_2}\right) \leq \beta + 0 + \beta = 2\beta.$$

Thus, we obtain that $\frac{\beta}{a^\top \bar{e}} \geq \frac{1}{2} > \frac{1}{3}$ in this case. Otherwise, we may choose $j \in [k]$ where $a_{j_1} > 0$. Consider the stable sets

$$S_0 \coloneqq ([k]_0 \setminus \{j_0\}) \cup \{j_1\}, S_1 \coloneqq ([k]_1 \setminus \{j_1\}) \cup \{j_0, j_2\}, S_2 \coloneqq ([k]_2 \setminus \{j_2\}) \cup \{j_1\}.$$

Now $\chi_{S_0} + \chi_{S_1} + \chi_{S_2} = \bar{e} + e_{j_1}$. Since $a_{j_1} > 0$, this implies that

$$a^\top \bar{e} < a^\top(\bar{e} + e_{j_1}) = a^\top \left(\chi_{S_0} + \chi_{S_1} + \chi_{S_2}\right) \leq 3\beta,$$

and so $\frac{\beta}{a^\top \bar{e}} > \frac{1}{3}$ in this case as well. □

We will also need the following result.

Lemma 30. [CCH89, Lemma 2.1] *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Given $u, v \in \mathbb{R}^n$ and positive real numbers $m_1, \ldots, m_d \in \mathbb{R}$, define*

$$x^{(i)} := u - \left(\sum_{i=1}^{d} \frac{1}{m_i}\right) v$$

for all $i \in [d]$. Suppose

(i) *$u \in P$, and*
(ii) *for all $i \in [d]$, $a^\top x^{(i)} \leq \beta$, for every inequality $a^\top x \leq \beta$ that is valid for P_I and satisfies $a \in \mathbb{Z}^n$ and $a^\top v < m_i$.*

Then $x^{(i)} \in \mathrm{CG}^i(P)$ for all $i \in [d]$.

We are now ready to prove Theorem 28.

Proof of Theorem 28. We first prove the rank lower bound. Given $d \geq 0$, let $k := \frac{1}{3}(2^{2d+1} + 7)$ (then $d = \log_4\left(\frac{3k-7}{2}\right)$). We show that the CG-rank of the inequality $\sum_{i \in B_{j,j'}} x_i \leq k - 1$ is at least $d + 1$ using Lemma 30.

Let $u := \frac{1}{2}\bar{e}, v := \bar{e}$, and $m_i := 2^{2i+1}$ for all $i \in [d]$. Then notice that $x^{(i)} = \frac{2^{2i+1}+1}{3 \cdot 2^{2i+1}}\bar{e}$ for all $i \in [d]$. Now suppose $a^\top x \leq \beta$ is valid for $\mathrm{STAB}(H_k)$ where a is an integral vector and $a^\top v < m_i$ (which translates to $a^\top \bar{e} < 2^{2i+1}$). Now Lemma 29 implies that $\frac{\beta}{a^\top \bar{e}} > \frac{1}{3}$. Furthermore, using the fact that $\beta, a^\top \bar{e}$ are both integers, $a^\top \bar{e} < 2^{2i+1}$, and $2^{2i+1} \equiv 2 \pmod 3$, we obtain that $\frac{\beta}{a^\top \bar{e}} \geq \frac{2^{2i+1}+1}{3 \cdot 2^{2i+1}}$, which implies that $a^\top x^{(i)} \leq \beta$. Thus, it follows from Lemma 30 that $x^{(i)} \in \mathrm{CG}^i(H_k)$ for every $i \in [d]$.

In particular, we obtain that $x^{(d)} = \frac{2^{2d+1}+1}{3 \cdot 2^{2d+1}}\bar{e} \in \mathrm{CG}^d(H_k)$. However, notice that $x^{(d)}$ violates the inequality $\sum_{i \in B_{j,j'}} x_i \leq k - 1$ for $\mathrm{STAB}(H_k)$, as

$$\frac{k-1}{|B_{j,j'}|} = \frac{k-1}{3k-4} = \frac{2^{2d+1}+4}{3 \cdot 2^{2d+1}+3} > \frac{2^{2d+1}+1}{3 \cdot 2^{2d+1}}.$$

Next, we turn to proving the rank upper bound. Given $d \in \mathbb{N}$, let $k := 2^d + 1$ (then $d = \log_2(k-1)$). We prove that $\sum_{i \in B_{j,j'}} x_i \leq k - 1$ is valid for $\mathrm{CG}^d(H_k)$ by induction on d. When $d = 1$, we see that $k = 3$ and $B_{j,j'}$ induces a 5-cycle, so the claim holds.

Now assume $d \geq 2$, and $k = 2^d + 1$. Let j, j' be distinct, fixed indices in $[k]$. By the inductive hypothesis, if we let $T \subseteq [k] \setminus \{j, j'\}$ where $|T| = 2^{d-1} - 1$, then the inequality

$$x_{j_0} + x_{j_2'} + \sum_{\ell \in T} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq 2^{d-1} \tag{24}$$

is valid for $\mathrm{CG}^{d-1}(H_k)$ (since the subgraph induced by $\{\ell_0, \ell_1, \ell_2 : \ell \in T\} \cup \{j_0, j_2'\}$ is a copy of that by $B_{j,j'}$ in H_{k-1}). Averaging the above inequality over all possible choices of T, we obtain that

$$x_{j_0} + x_{j_2'} + \frac{2^{d-1}-1}{k-2} \sum_{\ell \in [k] \setminus \{j,j'\}} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq 2^{d-1} \tag{25}$$

is valid for $\mathrm{CG}^{d-1}(H_k)$. Next, using (24) plus two edge inequalities, we obtain that for all $T \subseteq [k] \setminus \{j, j'\}$ where $|T| = 2^{d-1} + 1$, the inequality

$$\sum_{\ell \in T} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq 2^{d-1} + 2$$

is valid for $\mathrm{CG}^{d-1}(H_k)$. Averaging the above inequality over all choices of T, we obtain

$$\frac{2^{d-1}+1}{k-2} \sum_{\ell \in [k] \setminus \{j,j'\}} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq 2^{d-1} + 2. \tag{26}$$

Taking the sum of (25) and $\frac{k-2^{d-1}-1}{2^{d-1}+1}$ times (26), we obtain that

$$x_{j_0} + x_{j_2'} + \sum_{\ell \in [k] \setminus \{j,j'\}} (x_{\ell_0} + x_{\ell_1} + x_{\ell_2}) \leq \frac{k - 2^{d-1} - 1}{2^{d-1}+1}(2^{d-1}+2) + 2^{d-1} \tag{27}$$

is valid for $\mathrm{CG}^{d-1}(H_k)$. Now observe that the left hand side of (27) is simply $\sum_{i \in B_{j,j'}} x_i$. On the other hand, the right hand side simplifies to $k-2+\frac{k}{2^{d-1}+1}$. Since $k = 2^d+1, 1 < \frac{k}{2^{d-1}+1} < 2$, and so the floor of the right hand side of (27) is $k-1$. This shows that the inequality $\sum_{i \in B_{j,j'}} x_i \leq k-1$ has CG-rank at most d. □

Thus, we conclude that the facet (3) has LS$_+$-rank $\Theta(|V(H_k)|)$ and CG-rank $\Theta(\log(|V(H_k)|))$. We remark that the two results are incomparable in terms of computational complexity since it is generally $\mathcal{NP}$-hard to optimize over $\mathrm{CG}^k(P)$ even for $k = O(1)$. These rank bounds for H_k also provides an interesting contrast with the aforementioned example involving line graphs of odd cliques from [ST99], which have LS$_+$-rank $\Theta(\sqrt{|V(G)|})$ and CG-rank $\Theta(\log(|V(G)|))$. In the context of the matching problem, odd cliques have CG-rank one with respect to the fractional matching polytope.

6. Symmetric graphs with high LS$_+$-ranks

So far we have established that there exists a family of graphs (e.g., $\{H_k : k \geq 2\}$) which have LS$_+$-rank $\Theta(|V(G)|)$. However, the previous best result in this context $\Theta(\sqrt{|V(G)|})$ was achieved by a vertex-transitive family of graphs (line graphs of odd cliques). In this section, we show that there exists a family of vertex-transitive graphs which also have LS$_+$-rank $\Theta(|V(G)|)$.

6.1. **The L_k construction.** In this section, we look into a procedure that is capable of constructing highly symmetric graphs with high LS$_+$-rank by virtue of containing H_k as an induced subgraph.

Definition 31. Given a graph G and an integer $k \geq 2$, define the graph $L_k(G)$ such that $V(L_k(G)) := \{i_p : i \in [k], p \in V(G)\}$, and vertices i_p, j_q are adjacent in $L_k(G)$ if

- $i = j$ and $\{p, q\} \in E(G)$, or
- $i \neq j$, $p \neq q$, and $\{p, q\} \notin E(G)$.

For an example, let C_4 be the 4-cycle with $V(C_4) := \{0, 1, 2, 3\}$ and

$$E(C_4) := \{\{0,1\},\{1,2\},\{2,3\},\{3,0\}\}.$$

Figure 4 illustrates the graphs $L_2(C_4)$ and $L_3(C_4)$.

Moreover, notice that if we define P_2 to be the graph which is a path of length 2, with $V(P_2) := \{0, 1, 2\}$ and $E(P_2) := \{\{0,1\},\{1,2\}\}$, then $L_k(P_2) = H_k$ for every $k \geq 2$. Thus, we obtain the following.

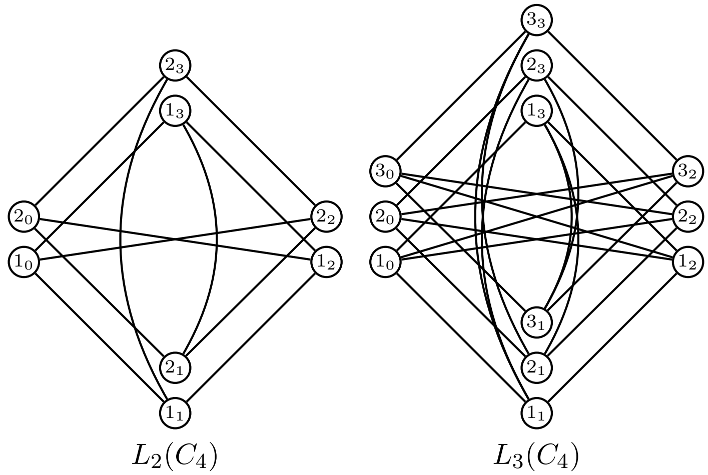

FIGURE 4. Illustrating the $L_k(G)$ construction on the 4-cycle

Proposition 32. *Let G be a graph that contains P_2 as an induced subgraph. Then the* LS_+*-rank lower bound in Theorem 27 for H_k also applies for $L_k(G)$.*

Proof. Since G contains P_2 as an induced subgraph, there must exist vertices $a, b, c \in V(G)$ where $\{a,b\}, \{b,c\} \in E(G)$, and $\{a,c\} \notin E(G)$. Then the subgraph of $L_k(G)$ induced by the vertices $\{i_p : i \in [k], p \in \{a,b,c\}\}$ is exactly $L_k(P_2) = H_k$. Thus, it follows from Lemma 5 that $r_+(L_k(G)) \geq r_+(H_k)$. $\square$

Since $L_k(C_4)$ has $4k$ vertices, Theorem 27 and Proposition 32 immediately imply the following.

Theorem 33. *Let $k \geq 3$ and $G \coloneqq L_k(C_4)$. Then $r_+(G) \geq \frac{1}{22}|V(G)|$.*

Since $\{L_k(C_4) : k \geq 3\}$ is a family of vertex-transitive graphs, Theorem 33 can also be proved directly by utilizing versions of the techniques in Sections 3 and 4. The graphs $L_k(C_4)$ are particularly noteworthy because C_4 is the smallest vertex-transitive graph that contains P_2 as an induced subgraph. In general, observe that if G is vertex-transitive, then so is $L_k(G)$. Thus, we now know that there exists a family of vertex-transitive graphs G with $r_+(G) = \Theta(|V(G)|)$.

6.2. **Generalizing the L_k construction.** Next, we study one possible generalization of the aforementioned L_k construction, and mention some interesting graphs it produces.

Definition 34. Given graphs G_1, G_2 on the same vertex set V, and integer $k \geq 2$, define $L_k(G_1, G_2)$ to be the graph with vertex set $\{i_p : i \in [k], p \in V\}$. Vertices i_p, j_q are adjacent in $L_k(G_1, G_2)$ if

- $i = j$ and $\{p, q\} \in E(G_1)$, or
- $i \neq j$ and $\{p, q\} \in E(G_2)$.

Thus, when $G_2 = \overline{G_1}$ (the complement of G_1), then $L_k(G_1, G_2)$ specializes to $L_k(G_1)$. Next, given $\ell \in \mathbb{N}$ and $S \subseteq [\ell]$, let $Q_{\ell,S}$ denote the graph whose vertices are the 2^ℓ binary strings of length ℓ, and two strings are joined by an edge if the number of positions they differ by is contained in S. For example, $Q_{\ell,\{1\}}$ gives the ℓ-cube. Then we have the following.

Proposition 35. *For every $\ell \geq 2$,*

$$L_4(Q_{\ell,\{1\}}, Q_{\ell,\{\ell\}}) = Q_{\ell+2,\{1,\ell+2\}}.$$

Proof. Let $G \coloneqq L_4(Q_{\ell,\{1\}}, Q_{\ell,\{\ell\}})$. Given $i_p \in V(G)$ (where $i \in [4]$ and $p \in \{0,1\}^{\ell}$), we define the function

$$f(i_p) \coloneqq \begin{cases} 00p & \text{if } i = 1; \\ 01\overline{p} & \text{if } i = 2; \\ 10\overline{p} & \text{if } i = 3; \\ 11p & \text{if } i = 4. \end{cases}$$

Note that $\overline{p}$ denotes the binary string obtained from p by flipping all ℓ bits. Now we see that $\{i_p, j_q\} \in E(G)$ if and only if $f(i_p)$ and $f(j_q)$ differ by either 1 bit or all $\ell + 2$ bits, and the claim follows. □

The graph $Q_{k,\{1,k\}}$ is known as the folded-cube graph, and Proposition 35 implies the following.

Corollary 36. *Let $G \coloneqq Q_{k,\{1,k\}}$ where $k \geq 3$. Then $r_+(G) \geq 2$ if k is even, and $r_+(G) = 0$ if k is odd.*

Proof. First, observe that when k is odd, $Q_{k,\{1,k\}}$ is bipartite, and hence has LS$_+$-rank 0. Next, assume $k \geq 4$ is even. Notice that $Q_{k-2,\{1\}}$ contains a path of length $k-2$ from the all-zeros vertex to the all-ones vertex, while $Q_{k-2,\{k-2\}}$ joins those two vertices by an edge. Thus, $Q_{k,\{1,k\}} = L_4(Q_{k-2,\{1\}}, Q_{k-2,\{k\}})$ contains the induced subgraph $L_4(P_{k-2})$ (where P_{k-2} denotes the graph that is a path of length $k-2$). Since $k-2$ is even, we see that $L_4(P_{k-2})$ can be obtained from $L_4(P_2) = H_4$ by odd subdivision of edges (i.e., replacing edges by paths of odd lengths). Thus, it follows from [LT03, Theorem 16] that $r_+(L_4(P_{k-2})) \geq 2$, and consequently $r_+(Q_{k,\{1,k\}}) \geq 2$. □

Example 37. The case $k = 4$ in Corollary 36 is especially noteworthy. In this case $G \coloneqq Q_{4,\{1,4\}}$ is the (5-regular) Clebsch graph. Observe that $G \ominus i$ is isomorphic to the Petersen graph (which has LS$_+$-rank 1) for every $i \in V(G)$. Thus, together with Corollary 36 we obtain that the Clebsch graph has LS$_+$-rank 2.

Alternatively, one can show that $r_+(G) \geq 2$ by using the fact that the second largest eigenvalue of G is 1. Then it follows from [ALT22, Proposition 8] that $\max\left\{\bar{e}^{\top}x : x \in \text{LS}_+(G)\right\} \geq 6$, which shows that $r_+(G) \geq 2$ since the largest stable set in G has size 5.

We remark that the Clebsch graph is also special in the following aspect. Given a vertex-transitive graph G, we say that G is *transitive under destruction* if $G \ominus i$ is also vertex-transitive for every $i \in V(G)$. As mentioned above, destroying any vertex in the Clebsch graph results in the Petersen graph, and so the Clebsch graph is indeed transitive under destruction. On the other hand, even though $L(G_1, G_2)$ is vertex-transitive whenever G_1, G_2 are vertex-transitive, the Clebsch graph is the only example which is transitive under destruction we could find using the L_k construction. For instance, one can check that $Q_{k,\{1,k\}} \ominus i$ is not a regular graph for any $k \geq 5$. Also, observe that the Clebsch graph can indeed be obtained from the "regular" L_k construction defined in Definition 31, as

$$Q_{4,\{1,4\}} = L_4(Q_{2,\{1\}}, Q_{2,\{2\}}) = L_4(C_4, \overline{C_4}) = L_4(C_4).$$

However, one can check that $L_k(C_\ell)$ is transitive under destruction if and only if $(k,\ell) = (4,4)$ (i.e., the Clebsch graph example), and that $L_k(K_{\ell,\ell})$ is transitive under destruction if and only if $(k,\ell) = (4,2)$ (i.e., the Clebsch graph example again). It would be fascinating to see what other interesting graphs can result from the L_k construction.

7. Some Future Research Directions

In this section, we mention some follow-up questions to our work in this manuscript that could lead to interesting future research.

Problem 38. What is the exact LS_+-rank of H_k?

While we showed that $r_+(H_k) \geq 0.19k$ asymptotically in Section 3, there is likely room for improvement for this bound. First, Lemma 19 is not sharp. In particular, the assumptions needed for $Y(e_0 - e_{1_0}), Y(e_0 - e_{1_1}) \in \text{LS}_+^{p-1}(H_k)$ are sufficient but not necessary. Using CVX, a package for specifying and solving convex programs [GB14, GB08] with SeDuMi [Stu99], we obtained that $r_+(H_6) \geq 3$. However, there do not exist a, b, c, d that would satisfy the assumptions of Lemma 19 for $k = 6$.

Even so, using Lemma 24 and the approach demonstrated in Example 25, we found computationally that $r_+(H_k) > 0.25k$ for all $k \leq 10000$. One reason for the gap between this computational bound and the analytical bound given in Theorem 27 is that the analytical bound only takes advantage of squeezing ℓ_i's over the interval $(u_1(k), u_2(k))$. Since we were able to show that $h(k, \ell) = \Theta(\frac{1}{k})$ over this interval (Lemma 26), this enabled us to establish a $\Theta(k)$ rank lower bound. Computationally, we see that we could get more ℓ_i's in over the interval $(u_2(k), u_3(k))$. However, over this interval, $h(k, \ell)$ is an increasing function that goes from $\frac{2}{k-2}$ at $u_2(k)$ to $\Theta\left(\frac{1}{\sqrt{k}}\right)$ at $u_3(k)$. This means that simply bounding $h(k, \ell)$ from above by $h(k, u_3(k))$ would only add an additional factor of $\Theta(\sqrt{k})$ in the rank lower bound. Thus, improving the constant factor in Theorem 27 would seem to require additional insights.

As for an upper bound on $r_+(H_k)$, we know that $r_+(H_4) = 2$, and $r_+(H_{k+1}) \leq r_+(H_k) + 1$ for all k. This gives the obvious upper bound of $r_+(H_k) \leq k - 2$. It would be interesting to obtain sharper bounds or even nail down the exact LS_+-rank of H_k.

Problem 39. Given $\ell \in \mathbb{N}$, is there a graph G with $|V(G)| = 3\ell$ and $r_+(G) = \ell$?

Theorem 6 raises the natural question: Are there graphs on 3ℓ vertices that have LS_+-rank exactly ℓ? Such ℓ-minimal graphs have been found for $\ell = 2$ [LT03] and for $\ell = 3$ [EMN06]. Thus, results from [LT03, EMN06] show that the answer is "yes" for $\ell = 1, 2, 3$. In [AT24], we construct the first 4-minimal graph which shows that the answer is also "yes" for $\ell = 4$. Does the pattern continue for larger ℓ? And more importantly, how can we verify the LS_+-rank of these graphs analytically or algebraically, as opposed to primarily relying on specific numerical certificates?

Problem 40. What can we say about the lift-and-project ranks of graphs for other positive semidefinite lift-and-project operators? To start with some concrete questions for this research problem, what are the solutions of Problems 38, 39 when we replace LS_+ with $\text{Las}, \text{BZ}_+, \Theta_p$, or SA_+?

After LS_+, many stronger semidefinite lift-and-project operators (such as Las [Las01], BZ_+ [BZ04], Θ_p [GPT10], and SA_+ [AT16]) have been proposed. While these stronger operators are capable of producing tighter relaxations than LS_+, these SDP relaxations can also be more computationally challenging to solve. For instance, while the LS_+^p-relaxation of a set $P \subseteq [0,1]^n$ involves $O(n^p)$ PSD constraints of order $O(n)$, the operators $\text{Las}^p, \text{BZ}_+^p$ and SA_+^p all impose one (or more) PSD constraint of order $\Omega(n^p)$ in their formulations. It would be interesting to determine the corresponding properties of graphs which are minimal with respect to these stronger lift-and-project operators.

Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

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