A COMPUTATIONAL SEARCH FOR MINIMAL OBSTRUCTION GRAPHS FOR THE LOVÁSZ–SCHRIJVER SDP HIERARCHY

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Abstract. We study the lift-and-project relaxations of the stable set polytope of graphs generated by LS_+, the SDP lift-and-project operator devised by Lovász and Schrijver. In particular, we focus on searching for ℓ-minimal graphs, which are graphs on 3ℓ vertices whose stable set polytope has rank ℓ with respect to LS_+. These are the graphs which are the most challenging for the LS_+ operator according to one of the main complexity measures (smallest graphs with largest LS_+-rank). We introduce the notion of LS_+ certificate packages, which is a framework that allows for efficient and reliable verification of membership of points in LS_+-relaxations. Using this framework, we present numerical certificates which (combined with other results) show that there are at least 49 3-minimal graphs, as well as over 4000 4-minimal graphs. This marks a significant leap from the 14 3-minimal and 588 4-minimal graphs known before this work, with many of the newly-discovered graphs containing novel structures which helps enrich and recalibrate our understanding of ℓ-minimal graphs. Some of this computational work leads to interesting conjectures. We also find all of the smallest vertex-transitive graphs with LS_+-rank ℓ for every $\ell \leq 4$.

1. Introduction

We are interested in studying the lift-and-project relaxations of the stable set polytope of graphs generated by the LS_+ operator due to Lovász and Schrijver [LS91]. To better put the main goals and results of this manuscript into perspective, we need to first briefly introduce the LS_+ operator, as well as what is currently known about the lift-and-project rank of graphs for the stable set problem. We then describe the problems we aim to address in this work, and provide a roadmap for how this manuscript is organized at the end of this section.

1.1. The LS_+ operator.

Given a set $P \subseteq [0,1]^n$, let

$$\text{cone}(P) := \left\{ \begin{bmatrix} \lambda \\ \lambda x \end{bmatrix} : \lambda \geq 0, x \in P \right\}$$

denote the *homogenized cone* of P. Notice that $\text{cone}(P) \subseteq \mathbb{R}^{n+1}$, and we will index the new coordinate by 0. Next, let $[n]$ denote the set $\{1, 2, \ldots, n\}$ for every $n \in \mathbb{N}$, and $\mathbb{S}_+^n$ denote the set of real, symmetric $n \times n$ positive-semidefinite (PSD) matrices. Alternatively, we will also use the notation $M \succeq 0$ to denote that M is a PSD matrix. We also let e_i denote the i-th unit vector

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(which has entry 1 at position i and 0 otherwise), and let $\operatorname{diag}(M)$ denote the vector formed by the diagonal entries of a square matrix M. Then we define

$$\widehat{\mathrm{LS}}_+(P) := \left\{ Y \in \mathbb{S}_+^{n+1} : Ye_0 = \operatorname{diag}(Y), Ye_i, Y(e_0 - e_i) \in \operatorname{cone}(P) \ \forall i \in [n] \right\},$$

and

$$\mathrm{LS}_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \widehat{\mathrm{LS}}_+(P), Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix} \right\}.$$

Given a set $P \subseteq [0,1]^n$, let $P_I := \operatorname{conv}\{P \cap \{0,1\}^n\}$ denote the *integer hull* of P (i.e., P_I is the convex hull of all integral points in P). A fundamental property of LS_+ is that $P_I \subseteq \mathrm{LS}_+(P) \subseteq P$ for every $P \subseteq [0,1]^n$ (see, for instance, [AT24b, Lemma 3] for a simple proof). Moreover, if P is tractable (i.e., polynomial-time separable up to arbitrary precision), then so is $\mathrm{LS}_+(P)$. Next, define $\mathrm{LS}_+^0(P) := P$, and then recursively define $\mathrm{LS}_+^\ell(P) := \mathrm{LS}_+\left(\mathrm{LS}_+^{\ell-1}(P)\right)$ for every $\ell \geq 1$. Then $\mathrm{LS}_+^n(P) = P_I$ for every $P \subseteq [0,1]^n$ — i.e., it always takes LS_+ no more than n iterations to tighten a set contained in the n-dimensional hypercube to its integer hull. (The reader may refer to [LS91] for the proofs of these properties and further discussion about LS_+.)

1.2. **The stable set problem and LS_+-rank of graphs.** Given a simple, undirected graph $G := (V(G), E(G))$, a set of vertices $S \subseteq V(G)$ is a *stable set* in G if no two vertices in S are joined by an edge in G. We then define the *fractional stable set polytope* of G to be

$$\mathrm{FRAC}(G) := \left\{ x \in [0,1]^{V(G)} : x_i + x_j \leq 1 \ \forall \{i,j\} \in E(G) \right\},$$

and the *stable set polytope* of G to be $\mathrm{STAB}(G) := \mathrm{FRAC}(G)_I$. Observe that a $0,1$-vector belongs to $\mathrm{STAB}(G)$ if and only if it is the incidence vector of a stable set in G. Also, for convenience, we will write $\mathrm{LS}_+^\ell(G)$ instead of $\mathrm{LS}_+^\ell(\mathrm{FRAC}(G))$. As discussed above, $\mathrm{LS}_+^\ell(G)$ provides successively tighter convex relaxations for $\mathrm{STAB}(G)$ as ℓ increases. This naturally leads to the notion of the LS_+-*rank* of G, which is defined to be the smallest integer ℓ where $\mathrm{LS}_+^\ell(G) = \mathrm{STAB}(G)$. For convenience, we also use the notation $r_+(G)$ to represent the LS_+-rank of G. The notion of rank also applies to specific inequalities: Given a valid inequality $a^\top x \leq \beta$ of $\mathrm{STAB}(G)$, its LS_+-rank is the smallest integer ℓ for which it is valid for $\mathrm{LS}_+^\ell(G)$.

It is well-known that $\mathrm{FRAC}(G) = \mathrm{STAB}(G)$ if and only if G is bipartite, so these are the only graphs which have LS_+-rank 0. Next, given a vector $a \in \mathbb{R}^n$, let $\operatorname{supp}(a)$ denote the *support* of a (i.e., the set of indices i where $a_i \neq 0$). The following general property of LS_+ is helpful for describing graphs with LS_+-rank 1.

Lemma 1. [LS91, Lemma 1.5] *Let $P \subseteq [0,1]^n$, and consider an inequality $a^\top x \leq \beta$ where $a \geq 0$ and $\beta > 0$. If $a^\top x \leq \beta$ is valid for $\{x \in P : x_i = 1\}$ for every $i \in \operatorname{supp}(a)$, then it is valid for $\mathrm{LS}_+(P)$.*

It follows from Lemma 1 that many families of valid inequalities of $\mathrm{STAB}(G)$ are valid for $\mathrm{LS}_+(G)$. For example, we say that $K \subseteq V(G)$ is a *clique* in G if every pair of vertices in K is joined by an edge in G. It is easy to see that the *clique inequality* $\sum_{i \in K} x_i \leq 1$ is valid for $\mathrm{STAB}(G)$ for every clique $K \subseteq V(G)$, as every stable set can only contain at most one vertex in K. Observe that, by Lemma 1, clique inequalities are valid for $\mathrm{LS}_+(G)$. Hence, if we define

$$\mathrm{CLIQ}(G) := \left\{ x \in [0,1]^{V(G)} : \sum_{i \in K} x_i \leq 1 \text{ for every clique } K \subseteq V(G) \right\}$$

to be the *clique polytope* of a given graph G, then we have $\mathrm{LS}_+(G) \subseteq \mathrm{CLIQ}(G)$ for every graph G. Thus, it follows that *perfect graphs* (i.e., graphs G where $\mathrm{STAB}(G) = \mathrm{CLIQ}(G)$) have LS_+-rank at most 1. Likewise, one can use Lemma 1 to show that odd cycle, odd wheel, and odd antihole inequalities are also valid for $\mathrm{LS}_+(G)$. The graphs with LS_+-rank 1 are known as LS_+*-perfect* graphs in the literature. For progress towards a combinatorial characterization of these graphs, see [BENT13, BENT17, Wag22, BENW23].

Of course, in addition to characterizing graphs whose stable set problem is "easy" to solve for LS_+, it is also insightful to study the graphs which serve as the worst-case instances for LS_+. First, let K_n denote the *complete graph* on n vertices (we will often use $[n]$ as the vertex labels for K_n). It follows from the results in [ST99] that the line graph of $K_{2\ell+1}$ has LS_+-rank ℓ, giving the first known family of graphs with unbounded LS_+-rank. Subsequently, Lipták and the second author [LT03] proved the following fact.

Theorem 2. [LT03, Theorem 39] *For every graph G, $r_+(G) \leq \left\lfloor \frac{|V(G)|}{3} \right\rfloor$.*

In other words, if we let $n_+(\ell)$ denote the minimum number of vertices among graphs with LS_+-rank ℓ, then Theorem 2 implies that $n_+(\ell) \geq 3\ell$ for every $\ell \in \mathbb{N}$. Thus, we say that a graph is *ℓ-minimal* if $|V(G)| = 3\ell$ and $r_+(G) = \ell$. It is easy to see that K_3 is the unique 1-minimal graph. In 2003, Lipták and the second authored showed that there is indeed a 2-minimal graph ($G_{1,1}$ from Figure 1), while conjecturing that ℓ-minimal graphs do exist for every $\ell \geq 1$. Subsequently, Escalante, Montelar, and Nasini [EMN06] showed that $G_{1,2}$ from Figure 1 is the only other 2-minimal graph, as well as discovered the first known 3-minimal graph ($G_{1,3}$ from Figure 1).

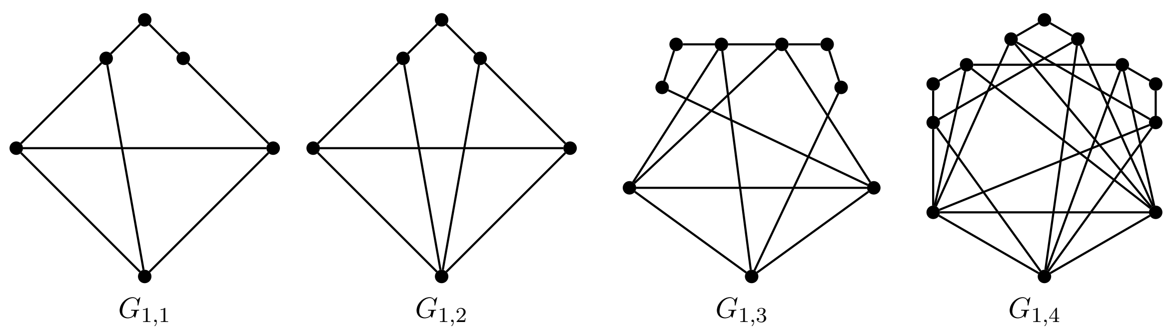


FIGURE 1. Several known ℓ-minimal graphs due to [LT03, EMN06, AT24a]

One common thread among these known ℓ-minimal graphs is that they can all be obtained from a complete graph by applying a number of vertex-stretching operations, which we describe below. Given a graph G and vertex $v \in V(G)$, define $\Gamma_G(v) := \{u : \{u, v\} \in E(G)\}$ to be the *(open) neighborhood* of v in G. Then given a vertex $v \in V(G)$ and sets $A_1, \ldots, A_k \subseteq \Gamma_G(v)$ where $\bigcup_{j=1}^{k} A_j = \Gamma_G(v)$, we define the *stretching* of v in G by applying the following sequence of transformations to G:

- replace v by $k + 1$ vertices: $v_0, v_1, \ldots, v_k$;
- for every $j \in [k]$, add an edge between v_j and all vertices in $\{v_0\} \cup A_j$.

We say that a vertex-stretching operation is *proper* if $\emptyset \neq A_j \subset \Gamma_G(v)$ for every $j \in [k]$. Also, we will call the operation *k-stretching* when we need to specify k. For example, in Figure 2, $G_1 = K_6$, G_2 is obtained from a proper 2-stretching of vertex 5 in G_1, and G_3 is obtained from a proper 2-stretching of vertex 6 in G_2.

An important property of the vertex-stretching operation in relation to the LS_+-rank of a graph is the following.

Theorem 3. [AT25b, Lemma 2] *Let H be obtained from G by stretching a vertex in G. Then $r_+(H) \geq r_+(G)$.*

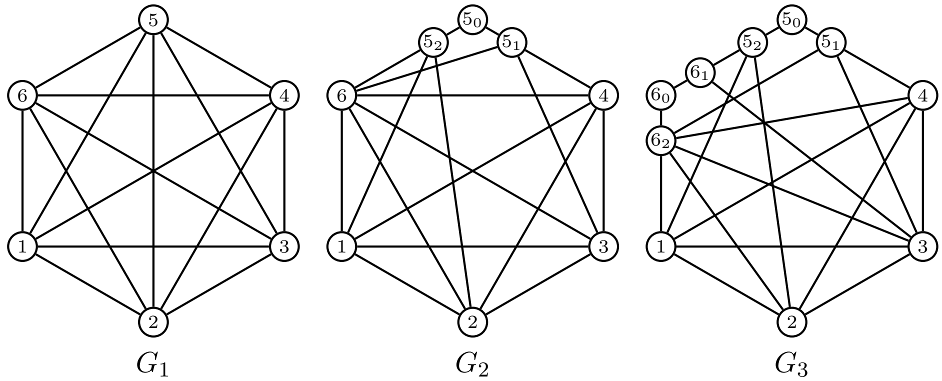


FIGURE 2. Illustrating the vertex-stretching operation

Theorem 3 is noteworthy because the LS$_+$-rank does not always behave well under the conventional graph minor operations. For instance, removing an edge from a graph could increase its LS$_+$-rank [LT03, Figure 4]. In fact, given a general graph G, there are currently only two known approaches to construct another graph H where $r_+(H) \geq r_+(G)$: Construct H so that G is an induced subgraph of H, or obtain H by (perhaps iteratively) stretching vertices in G.

While the vertex-stretching operation was first defined in its full generality in [AT25b], more restrictive variants of this operation had been previously studied. Notable examples include the type-1 stretching operation (which is a proper 2-stretching where $A_1 \cap A_2 = \emptyset$) and type-2 stretching operation (which is a proper k-stretching where $A_1, \ldots, A_k$ are mutually disjoint, with at least $k-1$ of these sets having size 1) studied by Lipták and the second author [LT03], who also proved that these two graph operations are LS$_+$-rank non-decreasing. These graph operations and other slight variants were also studied in other works — see, for instance, [AEF14, BENT17, AT24a].

Now, observe that the 2-minimal graphs $G_{1,1}$ and $G_{1,2}$ can each be obtained by a proper 2-stretching of a vertex in K_4. This gives examples where the LS$_+$-rank of a graph increases by 1 after a vertex-stretching operation. Likewise, $G_{1,3}$ can be obtained from taking K_5 and properly 2-stretching two of its vertices, increasing the graph's LS$_+$-rank from 1 to 3. On the other hand, we currently do not know of an instance where a single vertex-stretching operation increases the LS$_+$-rank of a graph by more than 1. This makes the special case of 2-stretching particularly attractive to study in terms of finding ℓ-minimal graphs, as it could increment the LS$_+$-rank of the graph while only increasing the number of vertices by two.

These insights motivated the authors' work in [AT24b, AT24a] In [AT24a], we studied the properties of the vertex-stretching operation, focusing on when the stretching is proper. In particular, the following result further highlights an important link between the 2-stretching operation and ℓ-minimal graphs.

Theorem 4. [AT24a, Theorem 19] *Let $\ell \geq 2$ be a positive integer, and let G be an ℓ-minimal graph. Then must exist a graph H and $i \in V(H)$ such that*

(i) *G is obtained from H by a proper 2-stretching of i, and*
(ii) *$H - i$ is an $(\ell - 1)$-minimal graph.*

We also showed the following when it comes to stretching vertices of a complete graph.

Theorem 5. [AT24a, Propositions 21 and 23] *Let $n \geq 4$ be an integer, and let G be obtained from K_n by a proper 2-stretching of $i \in V(K_n)$. Then*

(i) *$r_+(G) = 2$.*

(ii) *Let H be a graph obtained by a proper 2-stretching of one of $i_0, i_1, i_2 \in V(G)$. Then $r_+(H) = 2$.*

That is, while a proper 2-stretching of one of the original vertices $i \in V(K_n)$ is guaranteed to increase the graph's LS_+-rank from 1 to 2, the rank would remain at 2 if we further stretch any of the three new vertices i_0, i_1, or i_2. These results suggest that a promising approach of obtaining relatively small graphs with high LS_+-ranks is to 2-stretch some of the original vertices of K_n. Thus, given integers $n \geq 3$ and $d \geq 0$, let $\mathcal{K}_{n,d}$ denote the set of graphs which can be obtained from K_n by 2-stretching d of its vertices. Then notice that $G_{1,1}, G_{1,2} \in \mathcal{K}_{4,1}$, and $G_{1,3} \in \mathcal{K}_{5,2}$. We also discovered what was then the first known 4-minimal graph ($G_{1,4}$ from Figure 1) in [AT24a], which does belong to $\mathcal{K}_{6,3}$.

In addition to finding ℓ-minimal graphs, we also looked into the asymptotic behaviour of $n_+(\ell)$. In [AT24b], we constructed the family of graphs H_k as follows: For every integer $k \geq 3$, define

$$V(H_k) \coloneqq \{i_0, i_1, i_2 : i \in [k]\},$$
$$E(H_k) \coloneqq \{\{i_0, i_1\}, \{i_0, i_2\} : i \in [k]\} \cup \{\{i_1, j_2\} : i, j \in [k], i \neq j\}.$$

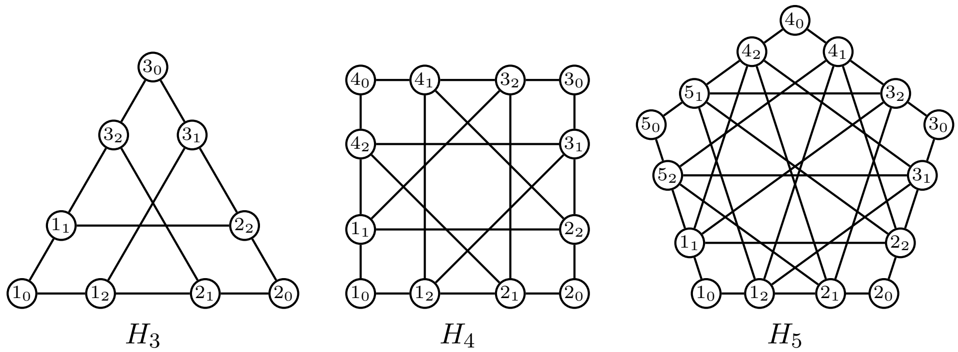


FIGURE 3. Several graphs in the family H_k

Figure 3 gives the drawings for H_k for $k \in \{3, 4, 5\}$. Then we showed the following.

Theorem 6. [AT24b, Theorem 2] *For every $k \geq 3$, $r_+(H_k) \geq \frac{3k}{16}$.*

The graphs H_k gave the first known family of graphs where $r_+(G) = \Theta(|V(G)|)$, which is asymptotically tight according to Theorem 2. Also, notice that the graphs H_k are highly symmetric. In particular, its automorphism group has only two orbits, which was conducive to the construction of an inductive argument which is based on showing that a two-dimensional "reduced" certificate satisfies the properties imposed by LS_+. Moreover, observe that $H_k \in \mathcal{K}_{k,k}$, which lends yet more credence to the idea that these stretched cliques are promising candidates for having high LS_+-rank.

More recently, the authors were able to further build on these ideas and prove that ℓ-minimal graphs indeed exist for every $\ell \in \mathbb{N}$. To describe these results, we need some more notation related to graphs that are stretched cliques. Given $G \in \mathcal{K}_{n,d}$, we let $D(G) \subseteq V(K_n)$ be the set of original vertices from K_n which were stretched to obtain G (and thus $|D(G)| = d$). Furthermore, for every $i \in V(K_n)$, we define the vertices in $V(G)$ *associated* with i to be i_0, i_1, and i_2 if $i \in D(G)$, and simply the unstretched vertex i otherwise. Observe that every vertex in G is associated with a unique original vertex in K_n. Also, by the definition of the vertex-stretching operation, observe that given $G \in \mathcal{K}_{n,d}$ and distinct $i, j \in [n]$, there must be at least one edge in G joining a vertex associated with i with a vertex associated with j.

Next, define $\hat{\mathcal{K}}_{n,d} \subseteq \mathcal{K}_{n,d}$ to be the set of stretched cliques G where, for every distinct $i, j \in D(G)$, there is *exactly* one edge in G that joins a vertex associated with i with a vertex associated with j. For example. observe that G_3 from Figure 2 does not belong to $\hat{\mathcal{K}}_{6,2}$, since $D(G_3) = \{5, 6\}$ and there are two edges — $\{5_1, 6_2\}$ and $\{5_2, 6_1\}$ — joining vertices associated with 5 and 6. On the other hand, notice that each of the four known ℓ-minimal graphs from Figure 1 belongs to $\hat{\mathcal{K}}_{\ell+2,\ell-1}$.

Given a graph G, let $\alpha(G)$ denote the size of the largest stable set in G. We also let $\bar{e}$ denote the vector of all-ones (the dimension of which will be clear from the context). For every $G \in \mathcal{K}_{n,d}$, we have $\alpha(G) = d+1$ [AT25b, Lemma 4], which implies that the inequality $\bar{e}^\top x \leq d+1$ is valid for $\mathrm{STAB}(G)$. Also, let $\omega(G)$ denote the size of the largest clique in G (usually known as the *clique number* of G). Then, we have the following.

Theorem 7. [AT25b, Theorem 19] *Let $G \in \hat{\mathcal{K}}_{n,d}$ where $n \geq 3$ and $d \geq 0$, and let $k := \max\{3, \omega(G)\}$. Then the LS_+-rank of $\bar{e}^\top x \leq d+1$ is at least $n-k+1$.*

Given $\ell \in \mathbb{N}$, observe that every $G \in \hat{\mathcal{K}}_{\ell+2,\ell-1}$ contains exactly 3ℓ vertices, and Theorem 7 assures that $r_+(G) = \ell$ as long as $\omega(G) \leq 3$. Using this, we were able to prove the following.

Theorem 8. [AT25b, Theorem 23] *For every positive integer ℓ, there are at least $2^{\ell-1}$ non-isomorphic ℓ-minimal graphs.*

While the bound in Theorem 8 is tight for $\ell = 1$ and $\ell = 2$, the number of ℓ-minimal graphs likely far exceeds $2^{\ell-1}$ for $\ell \geq 3$. For instance, there are 13 non-isomorphic graphs in $\hat{\mathcal{K}}_{5,2}$ with $\omega(G) \leq 3$ (see Figure 4), and it follows from Theorem 7 that they are all 3-minimal. (These 13 graphs do include $G_{1,3}$, as well as the 3-minimal graphs discovered earlier in [AT24a], up to isomorphism.) Moreover, there is at least one graph in $\mathcal{K}_{5,2} \setminus \hat{\mathcal{K}}_{5,2}$ which is also 3-minimal [AT25b, Proposition 25]. Likewise, an exhaustive computational search finds that there are 588 non-isomorphic graphs in $\hat{\mathcal{K}}_{6,3}$ with $\omega(G) \leq 3$, which include the first known 4-minimal graphs discovered in [AT24a]. Thus, at the time of this writing, there are a total of 14 known 3-minimal graphs and 588 known 4-minimal graphs.

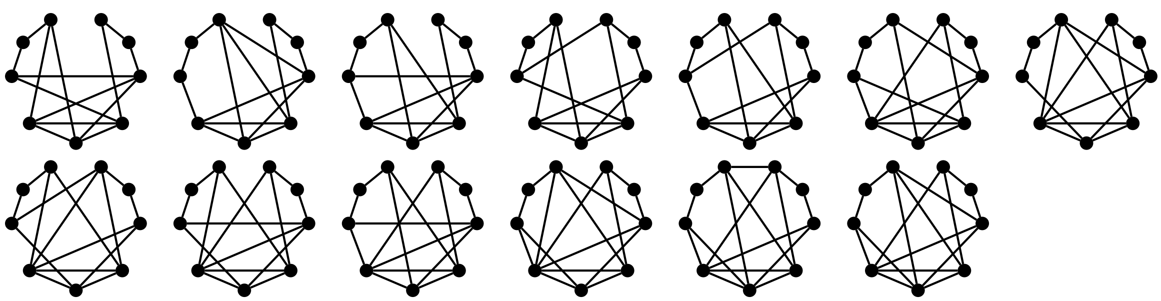


FIGURE 4. The 13 graphs $G \in \hat{\mathcal{K}}_{5,2}$ with $\omega(G) \leq 3$

Another consequence of Theorem 7 is the discovery of a family of vertex-transitive graphs with high LS_+-rank. Given an odd integer $k \geq 3$, the graph $\mathcal{B}_k$ is defined with

$$\begin{aligned} V(\mathcal{B}_k) &:= \{i_0, i_1, i_2, i_3 : i \in [k]\}, \\ E(\mathcal{B}_k) &:= \{\{i_0, i_1\}, \{i_1, i_2\}, \{i_2, i_3\}, \{i_3, i_0\} : i \in [k]\} \cup \\ &\quad \left\{\{i_0, j_2\}, \{i_1, j_3\} : (j-i) \bmod k \in \left\{1, 2, \ldots, \frac{k-1}{2}\right\}\right\}. \end{aligned}$$

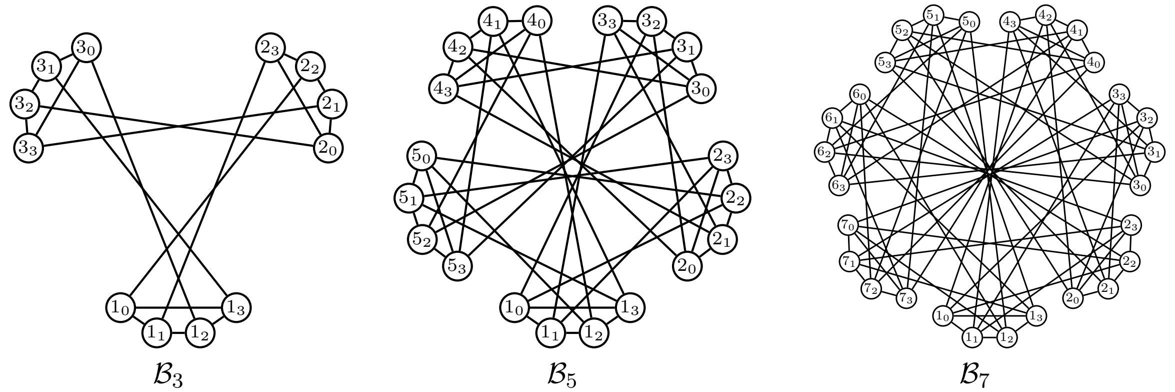


FIGURE 5. Illustrating the graphs $\mathcal{B}_k$

Figure 5 gives the drawings of $\mathcal{B}_k$ for $k \in \{3,5,7\}$. Notice that $\mathcal{B}_k$ is vertex-transitive, contains an induced subgraph in $\hat{\mathcal{K}}_{k,k}$ and has clique number 2 (see the proof of [AT25b, Proposition 26] for more details to these claims). Thus, it follows from Theorem 7 that $r_+(\mathcal{B}_k) \geq k-2$, and we have the following.

Theorem 9. *For every odd integer $\ell \geq 1$, there exists a vertex-transitive graph G where $|V(G)| \leq 4\ell + 8$ and $r_+(G) \geq \ell$.*

1.3. **Motivation and an outline for this manuscript.** With the recent results mentioned above, we have taken significant steps forward in our quest of understanding of LS_+-relaxations of the stable set polytopes of graphs. Most notably, we now know that $n_+(\ell) = 3\ell$ for every $\ell \in \mathbb{N}$, which completely settles the aforementioned conjecture by Lipták and the second author from 2003. On the other hand, new understanding naturally begets new questions. For instance, are there also ℓ-minimal graphs beyond those in $\hat{\mathcal{K}}_{\ell+2,\ell-1}$ described in Theorem 7? If so, what common structures do such graphs share? Must these graphs also have a small clique number? Painting a more complete picture of ℓ-minimal graphs would not only help us better understand the LS_+-relaxations of the stable set polytope of graphs, but these findings could also have implications on the convex relaxations of related problems in combinatorial optimization, which may lead to new understanding on the extension complexity of these problems. Also, by virtue of having the extremal property of being the smallest possible graphs with a given LS_+-rank, ℓ-minimal graphs could also possess other interesting properties that make them useful examples for other studies in graph theory as well as combinatorial optimization.

Thus, we are interested in performing a computational search for ℓ-minimal graphs, and seeing what this approach can uncover beyond what was found from previous analyses. Now, given a graph G on 3ℓ vertices, $r_+(G) \leq \ell$ follows readily from Theorem 2, and so the main challenge in proving that G is ℓ-minimal is to show that $r_+(G) \geq \ell$. To do so, the standard approach is to show that there exists $\bar{x} \in [0,1]^{V(G)}$ where $\bar{x} \in \text{LS}_+^{\ell-1}(G) \setminus \text{STAB}(G)$. In particular, showing $\bar{x} \in \text{LS}_+^{\ell-1}(G)$ for a specific graph G often involves presenting numerical certificates which satisfy the conditions imposed by LS_+. For example, during the last 25 years, the LS_+-rank lower-bound proofs for $G_{1,1}$ [LT03], $G_{1,2}$ [EMN06], $G_{1,3}$ [EMN06], and $G_{1,4}$ [AT24a] all followed this approach.

However, proofs involving such numerical certificates can be prone to both computational and human error (such as in the case of the initial proof for $r_+(G_{1,2}) \geq 2$, as pointed out in [AT24a]). For instance, computer algebra systems and SDP solvers can be numerical unstable (see for instance [WNM12]), which could lead to the errant conclusion that a matrix with a barely-negative eigenvalue is PSD. Also, the traditional print format could be less than ideal for presenting these certificates, as transferring this data from a computational software to print during the writing the process (and then vice versa if a reader would like to computationally verify this data) can be cumbersome, let alone serving as another opportunity for inaccuracies and errors to seep in. All of these issues are exacerbated as the dimensions of the certificates increase, especially for the cases $\ell \geq 3$ where we have to verify multiple layers of numerical certificates due to the recursive nature of the definition of $\mathrm{LS}_+^\ell(G)$.

Therefore, another main goal of this paper is to address these issues raised above by presenting a framework for a systemic presentation of numerical certificates for LS_+-relaxations. Under our proposed framework, verification of these certificates only depends on whole-number arithmetic, which can be carried out efficiently and reliably in computational algebra systems. We also believe that our framework for LS_+-relaxations can serve as a template for presenting and verifying certificates for other convex relaxations and for computational optimization in general. All numerical certificates presented in this paper are made publicly available [AT25a], so the interested reader could easily verify our work and perform their own analyses with this data.

The rest of the paper is organized as follows. In Section 2, we discuss the existing tools for proving LS_+-rank bounds. We also introduce the notion of LS_+ *certificate packages*, and provide an elementary example to illustrate the idea. Sections 3 and 4 are respectively dedicated to our search for 3-minimal and 4-minimal graphs. In particular, we show that there are at least 49 3-minimal graphs, and at least 4107 4-minimal graphs, which is a significant step forward from the 14 3-minimal graphs and 588 4-minimal graphs known at the time of this work. We point out the many interesting properties of these newfound examples, and put into perspective how their discovery augments and refines our understanding of ℓ-minimal graphs. We also go into details on our computational search for these graphs, as well as provide some relevant statistics.

We then turn our attention to vertex-transitive graphs in Section 5. In particular, we show that the smallest vertex-transitive graph with LS_+-rank 2, 3, and 4 have 8, 13, and 16 vertices respectively. Interestingly, these graphs share some structural similarities with stretched cliques. Finally, we conclude the manuscript by mentioning several future research directions in Section 6.

2. Tools for analyzing LS_+-ranks

In this section, we collect a number of tools which are useful in analyzing the LS_+-rank of a graph. We first mention several known relevant results, then describe a framework for presenting numerical certificates for LS_+ relaxations.

2.1. Some useful known tools.

First, the following is a well-known property of LS_+. (See, for instance, [AT24b, Lemma 5] for a proof.)

Lemma 10. *Let $P \subseteq [0,1]^n$ be a polyhedron, and F be a face of $[0,1]^n$. Then*

$$\mathrm{LS}_+^\ell(P \cap F) = \mathrm{LS}_+^\ell(P) \cap F,$$

for every $\ell \in \mathbb{N}$.

Next, given a graph G and a set of vertices $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. The following is an immediate consequence of Lemma 10. (See, for instance, [AT24b, Lemma 6] for a proof.)

Lemma 11. *Let G be a graph. Then $r_+(G[S]) \leq r_+(G)$ for every $S \subseteq V(G)$.*

Thus, the LS_+-rank of an induced subgraph of a graph cannot exceed that of the graph itself. Another implication of Lemma 10 is the following.

Lemma 12. *For every graph G,*

$$r_+(G) = \max\left\{ r_+(G[\text{supp}(a)] : a^\top x \leq \beta \text{ is a facet-inducing inequality of } \text{STAB}(G) \right\}.$$

Proof. ($\geq$) follows immediately from Lemma 11, so it only remains to prove ($\leq$). Let $\ell \coloneqq r_+(G)$. If $\ell = 0$, then G is bipartite and every induced subgraph must also have LS_+-rank 0. Thus, assume $\ell \geq 1$, and so there exists $\bar{x} \in \text{LS}_+^{\ell-1}(G) \setminus \text{STAB}(G)$, which means that there exists a facet-inducing inequality $a^\top x \leq \beta$ of $\text{STAB}(G)$ which is violated by $\bar{x}$. Then, it follows from Lemma 10 that the projection of $\bar{x}$ onto $\text{supp}(a)$ belongs to $\text{LS}_+^{\ell-1}(G[\text{supp}(a)]) \setminus \text{STAB}(G[\text{supp}(a)])$, and thus $r_+(G[\text{supp}(a)]) \geq \ell$. □

For the sake of brevity, we will slightly abuse terminology and refer to a facet-inducing inequality simply as a facet from here on. With Lemma 12, we see that if $\text{STAB}(G)$ does not have a full-support facet, then there exists a proper induced subgraph of G which has the same LS_+-rank as G. This immediately implies that the stable set polytope of an ℓ-minimal graph must have a full-support facet.

Another situation where one can conclude that $r_+(G)$ is realized by a proper subgraph of G is when the graph contains a *cut clique* — a clique whose removal from G results in multiple components. More precisely, we have the following.

Proposition 13. [LT03, Lemma 5] *Let G be a graph, and $S_1, S_2, K \subseteq V(G)$ are mutually disjoint subsets such that*

- *$S_1 \cup S_2 \cup K = V(G)$;*
- *K induces a clique in G;*
- *there is no edge $\{i, j\} \in E(G)$ where $i \in S_1, j \in S_2$.*

Then $r_+(G) = \max\{r_+(G[S_1 \cup K]), r_+(G[S_2 \cup K])\}$.

Next, given a graph G and $S \subseteq V(G)$, we define

$$G - S \coloneqq G[V(G) \setminus S],$$

and refer to $G - S$ as the graph obtained from G by the *deletion* of S. When $S = \{v\}$, we will simply write $G - v$ instead of $G - \{v\}$ for convenience. Given a vertex $v \in V(G)$, we also define

$$G \ominus v \coloneqq G - (\{v\} \cup \Gamma_G(v)),$$

and call $G \ominus v$ the graph obtained from G by the *destruction* of v. The following result relates the LS_+-rank of G to that of subgraphs of G obtained via the deletion or destruction of a vertex in G.

Theorem 14. *For every graph G,*

(i) [LS91, Corollary 2.16] $r_+(G) \leq \max\{r_+(G \ominus i) : i \in V(G)\} + 1$;
(ii) [LT03, Theorem 36] $r_+(G) \leq \min\{r_+(G - i) : i \in V(G)\} + 1$.

Finally, given graphs G and H where $V(H) = V(G)$ and $E(H) \subseteq E(G)$, we say that H is an *edge subgraph* of G. Observe that $\text{FRAC}(G) \subseteq \text{FRAC}(H)$ in this case. Also, it follows readily from the definition of LS_+ that the operator preserves containment (i.e., if $P \subseteq P'$, then $\text{LS}_+(P) \subseteq \text{LS}_+(P')$). Hence, we have the following.

Lemma 15. *Let H be an edge subgraph of G, and let ℓ be a nonnegative integer.*

(i) *If $a^\top x \leq \beta$ is valid for $\text{LS}_+^\ell(H)$, then $a^\top x \leq \beta$ is valid for $\text{LS}_+^\ell(G)$.*
(ii) *If $a^\top x \leq \beta$ is not valid for $\text{LS}_+^\ell(G)$, then $a^\top x \leq \beta$ is not valid for $\text{LS}_+^\ell(H)$.*

A useful implication of Lemma 15 is that, if we have a graph G and a valid inequality $a^\top x \leq \beta$ of STAB(G) with LS_+-rank ℓ, then every edge subgraph of G where $a^\top x \leq \beta$ is valid for its stable set polytope also has LS_+-rank at least ℓ.

2.2. **Introducing LS_+ certificate packages.** We now describe our framework of presenting numerical certificates for LS_+ relaxations in this manuscript. In this section, we focus on certifying the membership of a point in $\text{LS}_+^\ell(P)$ for the case $\ell = 1$, which will help prepare for our discussion of the cases where $\ell \geq 2$ in subsequent sections.

Given a symmetric matrix $Y \in \mathbb{Z}^{n \times n}$, we say that matrices U, V, W is a *UVW-certificate* of Y if

- the entries of U, V, and W are all integers;
- $W^\top \left(U^\top U + V\right) W = kY$ for some positive integer k;
- V is symmetric and diagonally dominant (i.e., $\sum_{j \neq i} |V_{ij}| \leq V_{ii}$ for all $i \in [n]$).

Then we have the following elementary fact.

Lemma 16. *Suppose $Y \in \mathbb{Z}^{n \times n}$ is a symmetric matrix. Then $Y \succeq 0$ if and only if Y has a UVW-certificate.*

Proof. Let $Y \succeq 0$ be given, and let $d := \text{rank}(Y)$. If $d = 0$, then Y is the matrix of all zeros, and in this case $W := I_n$ and U, V being the $n \times n$ matrix of all zeros would do. Next, assume that d is positive. Then there exists a symmetric matrix $Y' \in \mathbb{Z}^{d \times d}$ that is a principal submatrix of Y where $\text{rank}(Y') = d$. Hence, we can write $Y = W_1^\top Y' W_1$ for some rational matrix W_1. Next, let λ be the smallest eigenvalue of Y'. Observe that Y' is positive semidefinite (due to being a principal submatrix of Y) and has full rank, and thus Y' must be positive definite, which implies that $\lambda > 0$. Then $Y' - \lambda I_d \succeq 0$ and so there exists a real matrix U_0 where $Y' = U_0^\top U_0 + \lambda I_d$. Observe that λI_d is a positive multiple of the identity matrix, and so we can let U_1 be a rational approximation sufficiently close to U_0 such that $V_1 := Y' - U_1^\top U_1$ is diagonally dominant and has rational entries. Now, we have $Y = W_1^\top (U_1^\top U_1 + V_1) W_1$, and one can multiply the rational matrices U_1, V_1, W_1 by a suitable integer to obtain the desired integral matrices U, V, W.

Conversely, suppose Y has a UVW-certificate. Since $U^\top U \succeq 0$ for every U and $V \succeq 0$ (as a property of diagonally dominant matrices), we obtain that $U^\top U + V \succeq 0$, which implies that $Y = \frac{1}{k} W^\top (U^\top U + V) W \succeq 0$. □

The presence of a UVW-certificate allows us to easily and reliably verify the positive semidefiniteness of a given matrix by performing only elementary arithmetic operations involving whole numbers.

Next, recall that e_i denotes the i-th unit vector. Similarly, we let $f_i := e_0 - e_i$ (we will be using this notation exclusively when working in the space of cone(P) for a set $P \subseteq [0,1]^n$, so it will always be clear what the 0-th coordinate is). Given a graph G with n vertices, we define an LS_+ *certificate package* to be

- A matrix $Y \in \mathbb{Z}^{(n+1) \times (n+1)}$ where
  - $Y = Y^\top$ and $Ye_0 = \text{diag}(Y)$;
  - $Ye_i, Yf_i \in \text{cone}(\text{FRAC}(G))$ for every $i \in [n]$.
- A UVW-certificate for Y.

Notice that the presence of an LS_+ certificate package asserts that $Ye_0 \in \text{cone}(\text{LS}_+(G))$. Also, since FRAC($G$) is a rational polytope, the conditions $Ye_i, Yf_i \in \text{cone}(\text{FRAC}(G))$ can be verified using elementary arithmetic operations on whole numbers. LS_+ certificate packages are useful for helping establish that a given graph has LS_+-rank at least 2.

Proposition 17. *Let G be a graph. Then $r_+(G) \geq 2$ if and only if there exist a valid inequality $a^\top x \leq \beta$ for* $\mathrm{STAB}(G)$ *and an* LS_+ *certificate package (Y, U, V, W) for G such that $\left(-\beta, a^\top\right) Y e_0 > 0$.*

Proof. Suppose $r_+(G) \geq 2$. Then, there exists $\bar{x} \in \mathrm{LS}_+(G) \setminus \mathrm{STAB}(G)$ and a facet $a^\top x \leq \beta$ of $\mathrm{STAB}(G)$ such that $a^\top \bar{x} > \beta$. Then, by the definition of LS_+ and the density of rationals, there exists a positive semidefinite matrix $\tilde{Y}$ with rational entries satisfying the second and third conditions for an LS_+ certificate package and such that $\left(-\beta, a^\top\right) \tilde{Y} e_0 > 0$. By a suitable positive integer scaling of $\tilde{Y}$, we arrive at a positive semidefinite integral matrix Y satisfying all conditions for the existence of a LS_+ certificate package (Y, U, V, W) for G such that $\left(-\beta, a^\top\right) Y e_0 > 0$ (the existence of (U, V, W) satisfying the last condition of the LS_+ certificate package follows from Lemma 16).

Now, suppose there exist a valid inequality $a^\top x \leq \beta$ for $\mathrm{STAB}(G)$ and an LS_+ certificate package (Y, U, V, W) for G such that $\left(-\beta, a^\top\right) Y e_0 > 0$. Then, by the definition of LS_+ certificate package, $Y e_0 \in \mathrm{cone}(\mathrm{LS}_+(G))$. By assumption, $Y e_0$ violates a valid inequality for $\mathrm{cone}(\mathrm{STAB}(G))$. Therefore, $\mathrm{cone}(\mathrm{STAB}(G)) \subset \mathrm{cone}(\mathrm{LS}_+(G))$ which implies $r_+(G) \geq 2$. □

As our first example, consider the graph $G_{6,1}$ in Figure 6.

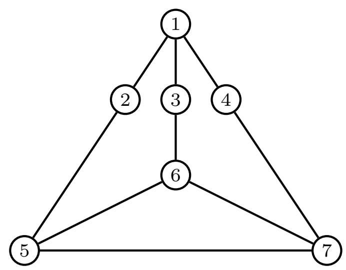


FIGURE 6. $G_{6,1}$, a 7-vertex graph with LS_+-rank 2

Then we have the following:

Proposition 18. *The graph $G_{6,1}$ from Figure 6 has* LS_+*-rank* 2.

Proof. For convenience, let $G := G_{6,1}$ throughout this proof. First, observe that $|V(G)| = 7$, and so $r_+(G) \leq 2$. Next, consider the matrices

$$Y := \begin{bmatrix} 76 & 25 & 40 & 40 & 40 & 20 & 20 & 20 \\ 25 & 25 & 0 & 0 & 0 & 10 & 10 & 10 \\ 40 & 0 & 40 & 30 & 30 & 0 & 10 & 10 \\ 40 & 0 & 30 & 40 & 30 & 10 & 0 & 10 \\ 40 & 0 & 30 & 30 & 40 & 10 & 10 & 0 \\ 20 & 10 & 0 & 10 & 10 & 20 & 0 & 0 \\ 20 & 10 & 10 & 0 & 10 & 0 & 20 & 0 \\ 20 & 10 & 10 & 10 & 0 & 0 & 0 & 20 \end{bmatrix}, \quad U := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 12 & -4 & -4 & 14 & -12 & -12 \\ 0 & 0 & -18 & 18 & 0 & -11 & 11 \\ 10 & 16 & 18 & 18 & -40 & -21 & -21 \\ 0 & 0 & 31 & -31 & 0 & -50 & 50 \\ 37 & 74 & -38 & -38 & -29 & 31 & 31 \\ 139 & 29 & 88 & 88 & 90 & 33 & 33 \end{bmatrix}$$

$$V := \begin{bmatrix} 185 & -17 & 30 & 30 & 17 & -16 & -16 \\ -17 & 183 & 20 & 20 & 8 & -11 & -11 \\ 30 & 20 & 227 & 37 & 34 & -44 & 12 \\ 30 & 20 & 37 & 227 & 34 & 12 & -44 \\ 17 & 8 & 34 & 34 & 303 & 17 & 17 \\ -16 & -11 & -44 & 12 & 17 & 264 & -14 \\ -16 & -11 & 12 & -44 & 17 & -14 & 264 \end{bmatrix}, \quad W := \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 \end{bmatrix}.$$

Then one can check that $6900Y = W^\top(U^\top U + V)W$, and the matrices indeed satisfy all conditions for an LS_+ certificate package. Thus, we obtain that

$$\bar{x} := \frac{1}{76}(25, 40, 40, 40, 20, 20, 20)^\top \in \text{LS}_+(G).$$

On the other hand, $\bar{x}$ violates the inequality $(2, 1, 1, 1, 1, 1, 1)^\top x \leq 3$, which is valid for $\text{STAB}(G)$. Thus, we conclude that $r_+(G) = 2$. □

The LS_+ certificate package described in the proof of Proposition 18, as well as other numerical data in support of the results in this manuscript, are made publicly available at [AT25a].

Notably, $G_{6,1}$ contains a *claw* (i.e., a stable set S of size 3 with a vertex that is adjacent to all 3 vertices in S). In fact, using the characterization of claw-free graphs with LS_+-rank at least 2 due to Bianchi et al. [BENW23, Corollary 32], it follows that if $|V(H)| = 7, r_+(H) = 2$ and that H does not contain $G_{1,1}$ or $G_{1,2}$ as an induced subgraph, then H must contain a claw. Thus, Proposition 18 shows that such an example indeed exists. The graph $G_{6,1}$ was also studied in [LT03], and Proposition 18 proves that the subdivision-of-a-star operation mentioned therein can increase the LS_+-rank of a graph.

Also, with Proposition 18, we can prove a slight generalization of Theorem 5(i).

Proposition 19. *Let $n \geq 4$, and let G be obtained from K_n by a proper stretching of a vertex. Then $r_+(G) = 2$.*

Proof. Suppose G is obtained from K_n by a proper k-stretching of $v \in V(K_n)$ for some integer $k \geq 2$. If $|A_i| \geq 2$ for at least one $i \in [k]$, then G must contain $G_{1,1}$ or $G_{1,2}$ as an induced subgraph. Otherwise, $|A_i| = 1$ for all $i \in [k]$, which implies that $k \geq 3$ (since $n \geq 4$). In this case, G must contains the graph $G_{6,1}$ as an induced subgraph. In either case, we have that $r_+(G) \geq 2$.

Also, notice that $G - v_0$ must be a perfect graph, and thus $r_+(G - v_0) \leq 1$, showing that $r_+(G) \leq 2$. □

Next, given a graph G, an integer $\ell \geq 1$ and a non-negative and non-zero vector $a \in \mathbb{R}^{V(G)}$, define

$$\gamma_\ell(G, a) := \frac{\max\left\{a^\top x : x \in \text{LS}_+^\ell(G)\right\}}{\max\left\{a^\top x : x \in \text{STAB}(G)\right\}}.$$

In other words, $\gamma_\ell(G, a)$ is the *integrality ratio* of $\text{LS}_+^\ell(G)$ in the direction of the vector a. By imposing that $a \geq 0$ and $a \neq 0$, we ensure that $\max\left\{a^\top x : x \in \text{STAB}(G)\right\} > 0$, and so $\gamma_\ell(G, a)$ is well-defined.

It is apparent that if $\gamma_\ell(G, a) > 1$ for some integer ℓ and vector a, then $r_+(G) > \ell$. Throughout this paper, we will establish LS_+-rank lower bounds of a graph using one of the following two approaches:

- Provide an analytical proof for this rank lower bound (e.g., using results stated earlier in this section);

- Present a point $\bar{x} \not\in \text{STAB}(G)$ with a $\text{LS}_+^{\ell-1}$ certificate package showing that $\bar{x} \in \text{LS}_+^{\ell-1}(G)$.

In both cases, we will write $r_+(G) \geq \ell$. Likewise, we write $r_+(G) \leq \ell$ if there is an analytical proof for this bound. On the other hand, there are situations when a self-contained argument using the existing theoretical tools is not available. In this case, we will try to obtain a "softer" upper bound using CVX+SeDuMi [GB14, Stu99], a MATLAB-based modelling system for convex optimization. In our experience — and especially in moderate to large size problem instances — it is not uncommon for CVX+SeDuMi to return an integrality ratio between $1 + 10^{-7}$ and $1 + 10^{-6}$ when there is an analytical proof that the true value is 1. Thus, given a graph G, we write that $r_+(G) \lesssim \ell$ if, for every facet $a^\top x \leq \beta$ of $\text{STAB}(G)$, either we have an analytical proof that $\gamma_\ell(G, a) = 1$, or

$$\gamma_\ell(G, a) \leq 1 + 10^{-6}$$

according to CVX+SeDuMi. In such cases, it is conceivable that the true value of $\gamma_\ell(G, a)$ is 1 for all facets of $\text{STAB}(G)$, which would imply that $r_+(G) \leq \ell$. Furthermore, due to Theorem 2, $\gamma_\ell(G, a) = 1$ for all facets $a^\top x \leq \beta$ of $\text{STAB}(G)$ where $|\operatorname{supp}(a)| < 3\ell$. Thus, to conclude that $r_+(G) \lesssim \ell$, it suffices to compute $\max\left\{a^\top x : x \in \text{LS}_+^\ell(G)\right\}$ with CVX+SeDuMi only for the facets where $|\operatorname{supp}(a)| \geq 3\ell$.

We remark that the computations for this work were mostly performed in MATLAB (R2023a) [Inc23] on a laptop computer equipped with an Intel Core i9-11950H processor (8 cores, 2.6 GHz clock speed) and 64 GB of RAM, running on the operating system Microsoft Windows 11 Education.

3. 3-minimal graphs

In this section, we focus on studying 3-minimal graphs (i.e., graphs G where $|V(G)| = 9$ and $r_+(G) = 3$). Again, at the time of this writing, the list of known 3-minimal graphs consists of the 13 in $\hat{\mathcal{K}}_{5,2}$ without a K_4 as an induced subgraph (Figure 4), as well as one other graph in $\mathcal{K}_{5,2} \setminus \hat{\mathcal{K}}_{5,2}$ [AT25b, Proposition 25].

Herein, we show that there are at least 49 non-isomorphic 3-minimal graphs in total, including 18 in $\mathcal{K}_{5,2} \setminus \hat{\mathcal{K}}_{5,2}$, and another 18 graphs which are not in $\mathcal{K}_{5,2}$. We also present some computational findings that could guide our search for ℓ-minimal graphs for $\ell \geq 4$.

To do so, we need to extend the notion of LS_+ certificate packages to consider an analogous framework for verifying the membership of points in $\text{LS}_+^2(G)$. First, we prove a simple lemma that helps explain one of the conditions in our LS_+^2 certificate packages. Given a set $P \subseteq [0,1]^n$, we say that P is *lower-comprehensive* if, for every $x \in P$, $0 \leq y \leq x$ implies $y \in P$. Observe that $\text{FRAC}(G)$ is lower-comprehensive for every graph G. It also follows readily from the definition of LS_+ that if P is lower-comprehensive, then so is $\text{LS}_+(P)$. Thus, we know that $\text{LS}_+^\ell(G)$ is lower-comprehensive for every graph G and every non-negative integer ℓ. Also, given $x^{(1)}, x^{(2)} \in \mathbb{R}^{n+1}$, we say that $x^{(1)}$ *dominates* $x^{(2)}$ if

- $x^{(1)} = x^{(2)} = 0$, or
- $[x^{(1)}]_0 > 0$, $[x^{(2)}]_0 \geq 0$, and $[x^{(2)}]_0 \cdot x^{(1)} \geq [x^{(1)}]_0 \cdot x^{(2)}$.

Then we have the following.

Lemma 20. *Let $P \subseteq [0,1]^n$ be a lower-comprehensive set, and let $x^{(1)}, x^{(2)} \in \mathbb{R}_+^{n+1}$. If $x^{(1)} \in \operatorname{cone}(P)$ and $x^{(1)}$ dominates $x^{(2)}$, then $x^{(2)} \in \operatorname{cone}(P)$.*

Proof. The claim obviously holds if $x^{(1)} = x^{(2)} = 0$, so we may assume that $[x^{(1)}]_0 > 0$, $[x^{(2)}]_0 \geq 0$, and $[x^{(2)}]_0 \cdot x^{(1)} \geq [x^{(1)}]_0 \cdot x^{(2)}$. Since $x^{(1)} \in \operatorname{cone}(P)$, we have $[x^{(2)}]_0 \cdot x^{(1)} \in \operatorname{cone}(P)$

(as cone(P) is closed under non-negative scalar multiplication). Also, given that P is lower-comprehensive, so is cone(P), and so it follows that $[x^{(1)}]_0 \cdot x^{(2)} \in \text{cone}(P)$. Using again the fact that cone(P) is closed under non-negative scalar multiplication (and that $[x^{(1)}]_0 > 0$), we conclude that $x^{(2)} \in \text{cone}(P)$. □

Next, given a graph G with n vertices, we define an LS_+^2 *certificate package* to be as follows:

- A set of matrices $\mathcal{M}_1 := \{Y_{e_i}, Y_{f_i} : i \in [n]\} \subseteq \mathbb{Z}^{(n+1)\times(n+1)}$ such that, for every $M \in \mathcal{M}_1$,
  - $M = M^\top$ and $Me_0 = \text{diag}(M)$;
  - $Me_i, Mf_i \in \text{cone}(\text{FRAC}(G))$ for every $i \in [n]$.
- A matrix $Y \in \mathbb{Z}^{(n+1)\times(n+1)}$ where
  - $Y = Y^\top$ and $Ye_0 = \text{diag}(Y)$;
  - for every $i \in [n]$,
    - $Y_{e_i}e_0$ dominates Ye_i;
    - $Y_{f_i}e_0$ dominates Yf_i.
- A UVW-certificate for every $M \in \mathcal{M}_1$ and Y.

Notice the conditions on the matrices in $\mathcal{M}_1$ certify that $Y_{e_i}e_0, Y_{f_i}e_0 \in \text{cone}(\text{LS}_+(G))$ for all $i \in [n]$. Next, using Lemma 20, the domination conditions assure that, for every $i \in [n]$,

$$Y_{e_i}e_0 \in \text{cone}(\text{LS}_+(G)) \Rightarrow Ye_i \in \text{cone}(\text{LS}_+(G)),$$
$$Y_{f_i}e_0 \in \text{cone}(\text{LS}_+(G)) \Rightarrow Yf_i \in \text{cone}(\text{LS}_+(G)).$$

Thus, together with other conditions on Y, we obtain that $Ye_0 \in \text{cone}(\text{LS}_+^2(G))$. Generally, a LS_+^2 certificate package for a vector in $\mathbb{R}^n$ consists of $4(1+2n)$ matrices (the certificate matrices $\mathcal{M}_1 \cup \{Y\}$, plus a UVW-certificate of each of these matrices). Due to the above arguments and following a similar proof to that of Proposition 17, we have the following fact.

Proposition 21. *Let G be a graph. Then $r_+(G) \geq 3$ if and only if there exist a valid inequality $a^\top x \leq \beta$ for* STAB(G) *and an* LS_+^2 *certificate package* ($Y, \mathcal{M}_1$, and UVW-certificates) *for G such that $\left(-\beta, a^\top\right) Ye_0 > 0$.*

We next show that there are at least 49 non-isomorphic 3-minimal graphs. First, Figure 7 gives eight of these graphs. Notice that every graph has the property that $\deg(8) = 2$, and the vertices $\{1, 2, 3, 4, 5, 6\}$ induce either $G_{1,1}$ or $G_{1,2}$.

Proposition 22. *Each of the eight graphs in Figure 7 is 3-minimal.*

Proof. First, for $i \in \{1, 2, 3\}$, $\bar{e}^\top x \leq 3$ is valid for STAB($G_{7,i}$), and for each of these graphs we provide an LS_+^2 certificate package [AT25a] for a point that violates this inequality, thus by Proposition 21, $r_+(G_{7,i}) \geq 3$.

Similarly, for $i \in \{4, 5, 6, 7, 8\}$, $(1, 1, 1, 1, 1, 1, 1, 1, 2)^\top x \leq 3$ is valid for STAB($G_{7,i}$), and we also provide LS_+^2 certificates packages for points violating this inequality.

Since $|V(G)| = 9$ readily implies $r_+(G) \leq 3$ (by Theorem 2), we obtain that $r_+(G) = 3$ in all eight cases. □

Next, it follows from Lemma 15 that if the inequality $a^\top x \leq \beta$ has LS_+-rank 3 for STAB(G) and is also valid for STAB(H) where H is an edge subgraph of G, then $r_+(H) \geq 3$. Thus, we obtain that many edge subgraphs of $G_{7,1}, \ldots, G_{7,8}$ are also 3-minimal. After removing isomorphic graphs, this results in a list of 49 3-minimal graphs, which we list in Figure 8. (Note that, to reduce cluttering and with every vertex having a single-digit vertex label, we use ij to denote the edge $\{i, j\}$.)

In particular, this proves the following.

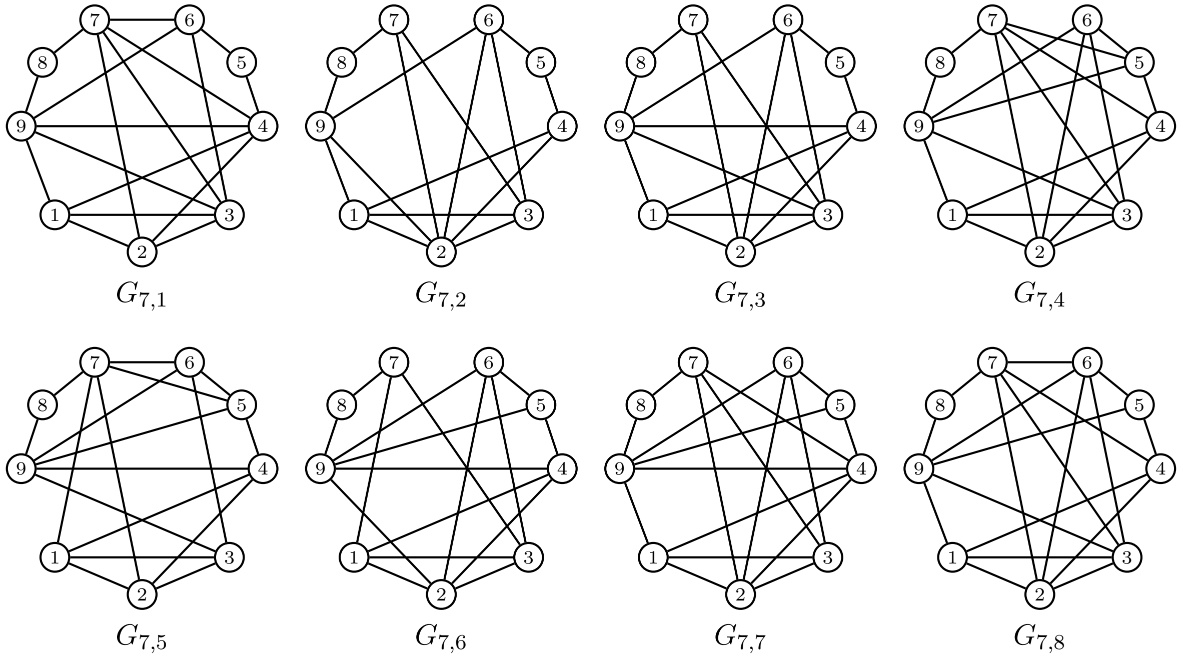


FIGURE 7. Eight edge-maximal 3-minimal graphs

Theorem 23. *There are at least* 49 *non-isomorphic* 3*-minimal graphs.*

We remark that the 49 graphs described in Figure 8 contain (up to isomorphism) every previously known 3-minimal graphs described in [EMN06, AT24b, AT25b]. Thus, for some of these graphs, we now have multiple independent proofs of them being 3-minimal.

Next, let us relate Theorem 23 to the existing findings about 3-minimal graphs. First, notice that $G_{7,1}$, $G_{7,2}$, $G_{7,3}$ and their edge subgraphs described in Figure 8 all belong to $\mathcal{K}_{5,2}$. 13 of these graphs further belong to $\hat{\mathcal{K}}_{5,2}$ and are exactly those shown in Figure 4. Our list also contains 18 3-minimal graphs which belong to $\mathcal{K}_{5,2} \setminus \hat{\mathcal{K}}_{5,2}$.

On the other hand, $G_{7,4}, \ldots, G_{7,8}$, as well as their edge subgraphs described in Figure 8, do not belong to $\mathcal{K}_{5,2}$. (One way to see this is that every graph $\mathcal{K}_{n,d}$ has at least d vertices of degree 2.) Thus, these graphs provide the first known instances of ℓ-minimal graphs which cannot be obtained by stretching the vertices of a clique. Also, with 19 edges, $G_{7,4}$ and $G_{7,5}$ are the densest known 3-minimal graphs yet. The sparsest possible 3-minimal graphs contain 14 edges [AT24a, Proposition 28], which is attained by several graphs in $\hat{\mathcal{K}}_{5,2}$.

Now recall Theorem 4, which assures that every ℓ-minimal graph can be obtained from an $(\ell-1)$-minimal graph by applying the following two graph operations:

- *1-Join*: Adding a new vertex and joining it to some (or all) vertices of the existing graph;
- 2*-stretch*: Applying a proper 2-stretching operation to one of the existing vertices of the graph.

Applying this observation iteratively, we obtain the following:

Corollary 24. *Let G be an ℓ-minimal graph for some positive integer ℓ. Then, there exists graphs $G_1, \ldots, G_\ell$ and $H_1, \ldots, H_{\ell-1}$ such that*

- *$G_1 = K_3$ and $G_\ell = G$;*
- *for every $i \in [\ell - 1]$, H_i can be obtained from G_i by adding a new vertex and joining it to some (or all) vertices of G_i;*

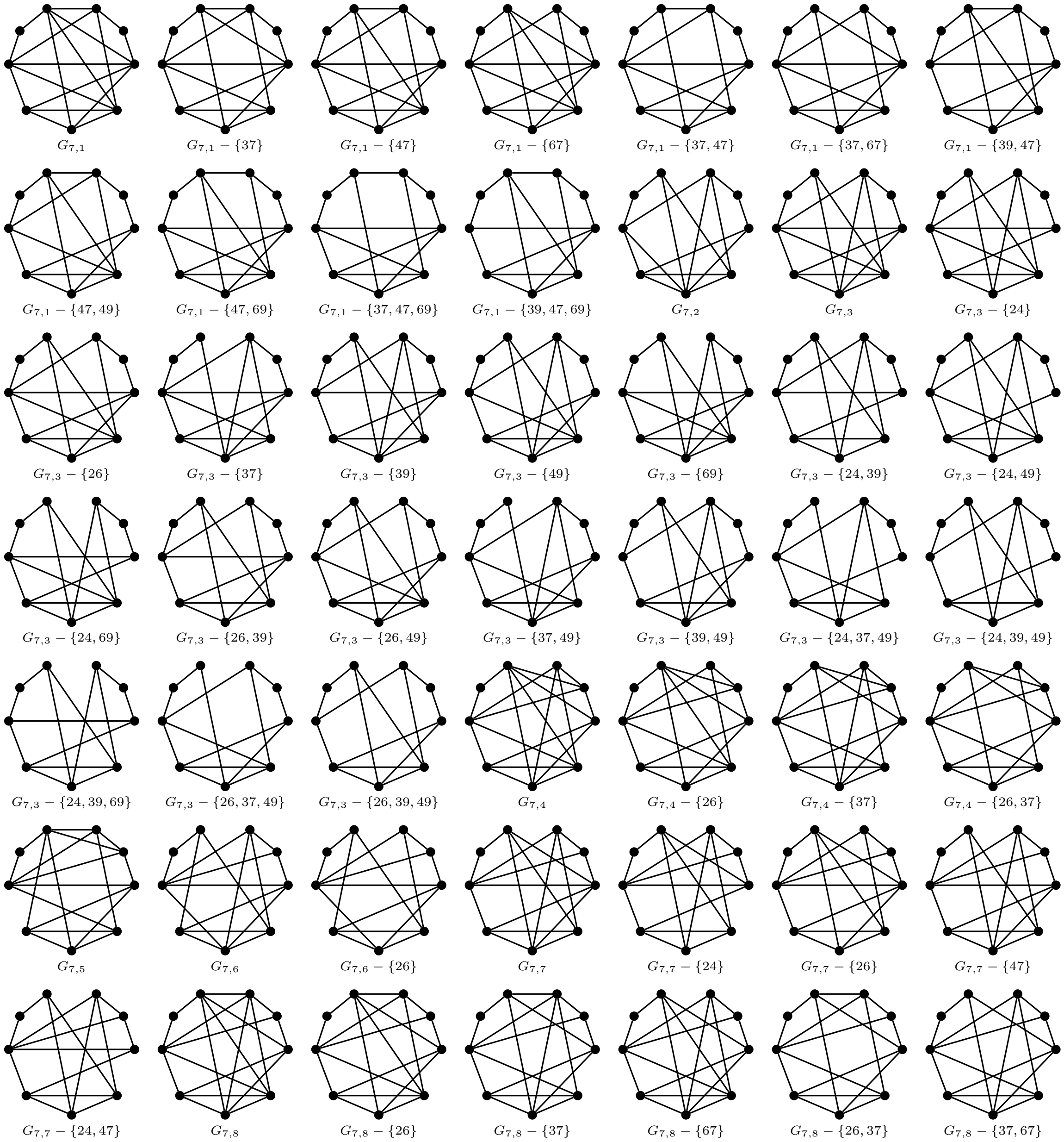


FIGURE 8. 49 non-isomorphic 3-minimal graphs

- *for every $i \in \{2, \ldots, \ell\}$, G_i can be obtained from H_{i-1} by a proper 2-stretching of a vertex.*

In other words, we see that every ℓ-minimal graph can be constructed by starting with K_3 (the unique 1-minimal graph), and then applying 1-join and 2-stretch operations alternatively. For some ℓ-minimal graphs, there is some flexibility in the sequencing of these 1-join and 2-stretch operations — for example, an ℓ-minimal graph in $\hat{\mathcal{K}}_{\ell+2,\ell-1}$ can be constructed by applying $\ell - 1$ 1-join operations in a row to K_3 to obtain $K_{\ell+2}$, and then applying $\ell - 1$ 2-stretch operations. However, the discovery of the 3-minimal graphs outside of $\mathcal{K}_{5,2}$ shows that, for some ℓ-minimal graphs, there is no flexibility in the order of these operations. For instance, Figure 9 shows the unique sequence of 1-join and 2-stretch operations with which we can construct $G_{7,4}$ from K_3.

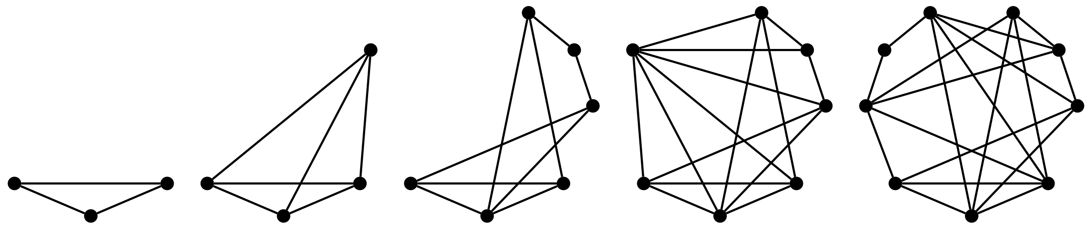


FIGURE 9. Obtaining $G_{7,4}$ from K_3 via 1-join and 2-stretch operations

Also, observe that a common feature of all 3-minimal graphs shown in Figure 8 is that they all contain K_5 as a graph minor. In particular, for each graph, contracting the edges $\{4,5\}$, $\{5,6\}$, $\{7,8\}$, and $\{8,9\}$ (and then removing parallel edges) would result in K_5. This further shows that each of the 49 graphs shown in Figure 8 contains a stretched clique in $\mathcal{K}_{5,2}$ as an edge subgraph. As we shall see in the next section, this pattern no longer holds for 4-minimal graphs.

Next, we turn our attention to the clique number of a graph, which has been shown in [AT25b] to be relevant in determining whether a graph is ℓ-minimal under some circumstances. First, we see that among the 49 3-minimal graphs shown in Figure 8, 48 of them have $\omega(G) = 3$, and one has $\omega(G) = 4$ ($G_{7,8}$, with the vertices $\{2,3,6,7\}$ inducing a K_4). Thus, we see that $\omega(G) \leq 3$ is not a necessary condition for a graph to be ℓ-minimal in general.

The situation seems to be more interesting if we restrict our discussion to stretched cliques. Before we go further, the following lemma will be helpful.

Lemma 25. *Given $G \in \mathcal{K}_{n,d}$ where $n \geq 3$ and $d \geq 0$, we have*

(i) $|E(G)| \geq \frac{n(n-1)}{2} + 2d$.

(ii) *If $\omega(G) \geq 3$, then G contains an edge subgraph $H \in \mathcal{K}_{n,d}$ where $|E(H)| = \frac{n(n-1)}{2} + 2d$ and $\omega(H) = \omega(G)$.*

Proof. We first prove (i). Given distinct $i.j \in [n]$, it follows from the definition of the vertex-stretching operation that there exists at least one edge in G which joins a vertex associated with i and a vertex associated with j, giving a total of at least $\frac{n(n-1)}{2}$ distinct edges. Also, for every $i \in D(G)$, we have the edges $\{i_0, i_1\}$ and $\{i_0, i_2\}$, which yields an additional total of $2|D(G)| = 2d$ edges. Thus, $|E(G)| \geq \frac{n(n-1)}{2} + 2d$.

For (ii), suppose $K \subseteq V(G)$ induces a clique of size at least three in G. Then observe that no two vertices in K can be associated with the same index in $[n]$. Thus, if there are multiple edges joining vertices associated with distinct $i, j \in [n]$, we can remove all but one of them without deleting edges that join vertices in K. Doing this for all distinct $i, j \in [n]$ would yield the desired graph H, which is an edge subgraph of G with $|E(H)| = \frac{n(n-1)}{2} + 2d$ and $\omega(H) = \omega(G)$ (since K still induces a clique in H). □

Therefore, given integers n, d where $n \geq 3$ and $d \geq 0$, we say that $G \in \mathcal{K}_{n,d}$ is a *sparse stretched clique* if $|E(G)| = \frac{n(n-1)}{2} + 2d$. Observe that a sparse stretched clique necessarily belongs to $\hat{\mathcal{K}}_{n,d}$.

Next, recall Theorem 7, which states that given $G \in \hat{\mathcal{K}}_{5,2}$, $\omega(G) \leq 3$ is sufficient for G to be 3-minimal. On the other hand, numerical evidence suggests that $\omega(G) \leq 3$ is not sufficient for

$G \in \mathcal{K}_{5,2}$ to be 3-minimal. In Figure 10, we list the seven graphs for which $\bar{e}^\top x \leq 3$ is the lone facet with full support, but computations from CVX+SeDuMi show that $r_+(G) \lesssim 2$ in all seven cases.

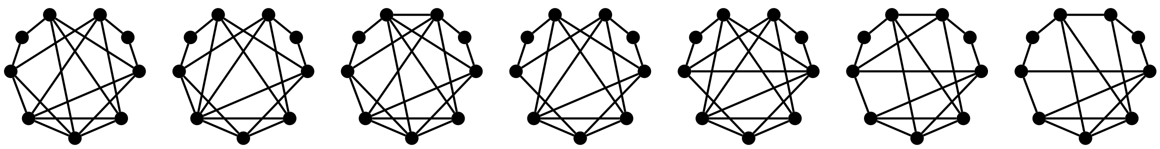


FIGURE 10. The seven graphs $G \in \mathcal{K}_{5,2}$ with $\omega(G) \leq 3$ and $r_+(G) \lesssim 2$

Thus, given $G \in \mathcal{K}_{5,2}$, the inequality $\bar{e}^\top x \leq 3$ may seem to have LS_+-rank two or three when $\omega(G) = 3$. On the other hand, we show that this inequality cannot have LS_+-rank 3 if $\omega(G) \geq 4$.

Proposition 26. *Let $G \in \mathcal{K}_{5,2}$. If $\omega(G) \geq 4$, then the inequality $\bar{e}^\top x \leq 3$ has LS_+-rank at most 2.*

Proof. Given $G \in \mathcal{K}_{5,2}$ with $\omega(G) \geq 4$, we know from Lemma 25 that G has an edge subgraph $H \in \mathcal{K}_{5,2}$ where $|E(H)| = 14$ and $\omega(H) = \omega(G)$. By Lemma 15, it suffices to show that $\bar{e}^\top x \leq 3$ has LS_+-rank at most 2 for $\mathrm{STAB}(H)$.

Next, let us focus on the graph H. If $\bar{e}^\top x \leq 3$ is not a facet of $\mathrm{STAB}(H)$, then it follows from the proof of [AT25b, Lemma 7] that $\bar{e}^\top x \leq 3$ can be expressed as the sum of facets of $\mathrm{STAB}(H)$ that do not have full support, in which case Lemma 12 implies that $r_+(H) \leq 2$. Thus, we may assume that $\bar{e}^\top x \leq 3$ is indeed a facet of $\mathrm{STAB}(H)$. (For a complete characterization of when $\bar{e}^\top x \leq d+1$ is a facet of the stable set polytope of a graph in $\mathcal{K}_{n,d}$, see [AT25b, Lemma 6].)

Now, an exhaustive search shows that there are exactly three non-isomorphic graphs $H \in \mathcal{K}_{5,2}$ where $|E(H)| = 14$, $\omega(H) \geq 4$, and $\bar{e}^\top x \leq 3$ is a facet of $\mathrm{STAB}(H)$, which are shown in Figure 11. We show that these three graphs all have LS_+-rank at most two. First, notice that $G_1 - 4$ is a perfect graph. Thus, $r_+(G_1 - 4) \leq 1$, which implies that $r_+(G_1) \leq 2$. Likewise, $G_2 - 3$ and $G_3 - 4$ are also perfect. Therefore, G_1, G_2, and G_3 (and thus, in particular, the inequality $\bar{e}^\top x \leq 3$ of their stable set polytopes) all have LS_+-rank at most 2. This proves our claim. ☐

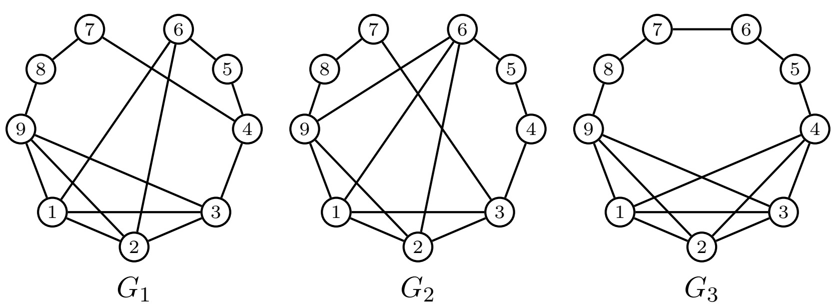


FIGURE 11. The three non-isomorphic graphs $H \in \mathcal{K}_{5,2}$ where $|E(H)| = 14$, $\omega(H) \geq 4$, and $\bar{e}^\top x \leq 3$ is a facet of $\mathrm{STAB}(H)$

Next, we provide some details about our computational search that turned up the 49 3-minimal graphs we listed in Figure 8. First, given a 3-minimal graph G, we may assume (due to Theorem 4) that G can be obtained from a proper 2-stretching of $i \in V(H)$ for some 7-vertex graph H where $H - i$ is isomorphic to $G_{1,1}$ or $G_{1,2}$ (which, again, are the only 2-minimal graphs).

An exhaustive search found 1115 non-isomorphic graphs which satisfy these conditions. Among them, there are 540 instances of (G, a) where $a \in \mathbb{R}^9$ defines a full-support facet for $\text{STAB}(G)$. For convenience, let $\mathcal{X}_3$ denote the set consisting of these 540 ordered pairs (G, a). Note that there are 9 graphs which appear in two elements in $\mathcal{X}_3$ for having two distinct full-support facets, and none of them belong to $\mathcal{K}_{5,2}$. If it is indeed true that no graph in $\mathcal{K}_{5,2}$ has a full-support facet that is different from $\bar{e}^\top x \leq 3$, then it would follow from Proposition 26 that $r_+(G) \leq 2$ for every $G \in \mathcal{K}_{5,2}$ where $\omega(G) \geq 4$.

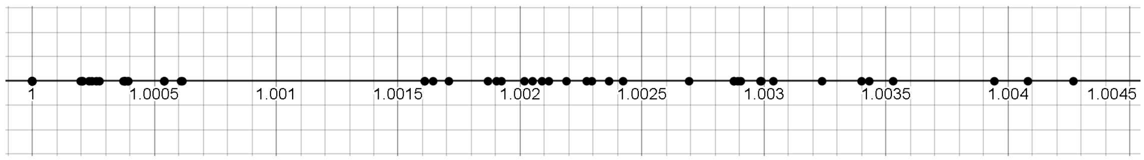


FIGURE 12. Plotting $\gamma_2(G, a)$ for all $(G, a) \in \mathcal{X}_3$

Using CVX+SeDuMi, we computed the values of $\gamma_2(G, a)$ for all 540 $(G, a) \in \mathcal{X}_3$, and plotted them on the number line as shown in Figure 12. The 540 data points can be categorized into the following three groups:

- 31 elements with $\gamma_2(G, a) > 1.0016$. They correspond exactly to $G_{7,1}$, $G_{7,2}$, $G_{7,3}$, and their edge subgraphs listed in Figure 8. These 31 graphs share the common property that they are all in $\mathcal{K}_{5,2}$ and have $\bar{e}^\top x \leq 3$ as the facet with LS_+-rank three.
- 18 elements with $1.00019 < \gamma_2(G, a) < 1.00062$. These correspond to $G_{7,4}, \ldots, G_{7,8}$ and their edge subgraphs listed in Figure 8. These 18 graphs do not belong to $\mathcal{K}_{5,2}$, and their facet with LS_+-rank three is $(1, 1, 1, 1, 1, 1, 1, 1, 2)^\top x \leq 3$.
- 491 elements with $\gamma_2(G, a) \leq 1 + 4 \cdot 10^{-7}$. They all visually correspond to the dot at 1 in Figure 12.

Thus, we see that aside from the 49 3-minimal graphs we previously described, every graph in our search satisfies $r_+(G) \lesssim 2$. This gives us a level of confidence that we have in fact found every 3-minimal graph, and that the list in Figure 8 is complete.

Conjecture 27. There are exactly 49 non-isomorphic 3-minimal graphs.

We conclude this section by presenting more computational findings for the graphs and facets in $\mathcal{X}_3$. Recall that we generated the collection $\mathcal{X}_3$ by exhaustively checking among a pool of candidate graphs for full-support facets. However, the approach of casting a wide net and searching exhaustively within may not be viable for 4-minimal graphs and beyond when both the number of candidate graphs and the time demand for each optimization problem increase substantially. Therefore, it is worthwhile to take a closer look at the numerical data in our search for 3-minimal graphs for possible insights that could guide our search for ℓ-minimal graphs for $\ell \geq 4$.

Thus, in addition to $\gamma_2(G, a)$, we also computed $\gamma_1(G, a)$ for every $(G, a) \in \mathcal{X}_3$, as we are interested to see if $\gamma_1(G, a)$ has some predictive value for $\gamma_p(G, a)$ for $p \geq 2$. If so, then $\gamma_1(G, a)$ (which is much less computationally costly to obtain) could serve as a valuable heuristic for identifying ℓ-minimal graphs.

Figure 13 (left) gives the scatterplot of $\gamma_1(G, a)$ (x-axis) versus $\gamma_2(G, a)$ (y-axis) for each of the 540 elements in $\mathcal{X}_3$. All $(G, a) \in \mathcal{X}_3$ satisfy $1.0036 \leq \gamma_1(G, a) \leq 1.0608$. It is not surprising that these integrality ratios are all comfortably above 1 since every graph analyzed here contains either $G_{1,1}$ or $G_{1,2}$ as an induced graph, and thus must have LS_+-rank at least 2. A simple linear regression shows a very weak ($r \approx 0.0947$) positive correlation between $\gamma_1(G, a)$ and $\gamma_2(G, a)$.

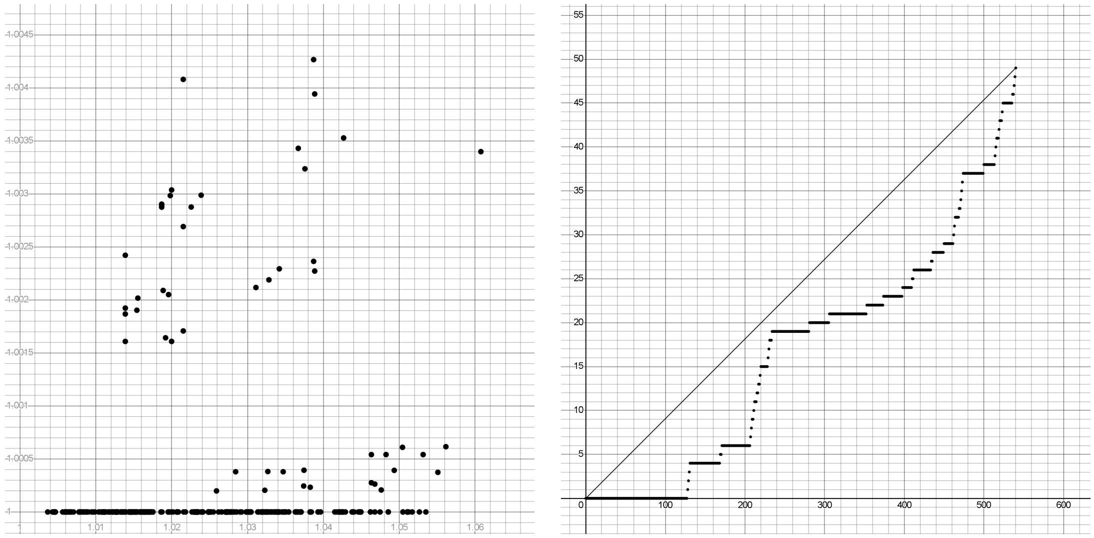


FIGURE 13. Relating $\gamma_1(G, a)$ and $\gamma_2(G, a)$ for elements in $\mathcal{X}_3$

Next, we analyze the relation between $\gamma_1(G, a)$ and $\gamma_2(G, a)$ from a different perspective. For every n where $0 \leq n \leq 540$, we define $g(n)$ to be the number of the 49 3-minimal graphs listed in Figure 8 we would find among the n facets with the lowest $\gamma_1(G, a)$. Obviously, $g(0) = 0$ and $g(540) = 49$, and Figure 13 (right) gives the plot of $g(n)$ for $0 \leq n \leq 540$. Observe that the points $(n, g(n))$ all stay below the line $y = \frac{49}{540}x$, which indicates that the facets with higher $\gamma_1(G, a)$ indeed contain a higher concentration of facets with LS_+-rank 3. In particular, notice that $g(n) = 0$ for all $n \leq 127$, which means that the 127 facets with the lowest values of $\gamma_1(G, a)$ all basically have $\gamma_2(G, a) = 1$.

4. 4-MINIMAL GRAPHS

We now turn our attention to 4-minimal graphs. As with LS_+ and LS_+^2 certificate packages, here is an analogous framework for verifying the membership of points in $\text{LS}_+^3(G)$. Given a graph G with n vertices, we define an LS_+^3 *certificate package* to be

- A set of matrices $\mathcal{M}_2 := \left\{ Y_{e_i e_j}, Y_{e_i f_j}, Y_{f_i e_j}, Y_{f_i f_j} : i, j \in [n] \right\} \subseteq \mathbb{Z}^{(n+1) \times (n+1)}$ such that, for every $M \in \mathcal{M}_2$,
  - $M = M^\top$ and $Me_0 = \text{diag}(M)$;
  - $Me_i, Mf_i \in \text{cone}(\text{FRAC}(G))$ for every $i \in [n]$.
- A set of matrices $\mathcal{M}_1 := \{ Y_{e_i}, Y_{f_i} : i \in [n] \} \subseteq \mathbb{Z}^{(n+1) \times (n+1)}$ such that, for every $M \in \mathcal{M}_1$,
  - $M = M^\top$ and $Me_0 = \text{diag}(M)$;
  - for every $i, j \in [n]$,
    - $Y_{e_i e_j} e_0$ dominates $Y_{e_i} e_j$;
    - $Y_{e_i f_j} e_0$ dominates $Y_{e_i} f_j$;
    - $Y_{f_i e_j} e_0$ dominates $Y_{f_i} e_j$;
    - $Y_{f_i f_j} e_0$ dominates $Y_{f_i} f_j$.
- A matrix $Y \in \mathbb{Z}^{(n+1) \times (n+1)}$ where
  - $Y = Y^\top$ and $Ye_0 = \text{diag}(Y)$;
  - for every $i \in [n]$,
    - $Y_{e_i} e_0$ dominates Ye_i;

    * $Y_{f_i} e_0$ dominates $Y f_i$.
- A UVW-certificate for every non-zero matrix in $\mathcal{M}_2, \mathcal{M}_1$, and Y.

The conditions on $\mathcal{M}_2$ establish that $Me_0 \in \text{cone}(\text{LS}_+(G))$ for every $M \in \mathcal{M}_2$. Then the conditions imposed on $\mathcal{M}_1$ assure that $Me_0 \in \text{cone}(\text{LS}_+^2(G))$ for every $M \in \mathcal{M}_1$. Finally, the additional constraints on Y ensure that $Ye_0 \in \text{cone}(\text{LS}_+^3(G))$. Now observe that $\mathcal{M}_2$ inherently contains many matrices of all zeros — for instance, for every edge $\{i, j\} \in E(G)$, LS_+ imposes the constraint $[Ye_i]_j = 0$. Hence, $Y_{e_i} e_j$ is the zero vector, and consequently $Y_{e_i e_j}$ must be a matrix of all zeros. Matrices of all zeros are trivially PSD, and not including UVW-certificates for them helps somewhat reduce the size of these certificate packages. Thus, a LS_+^3 certificate package for a vector in $\mathbb{R}^n$ consists of up to $4(1 + 2n + 4n^2)$ matrices (the certificate matrices $\mathcal{M}_2, \mathcal{M}_1$, and Y, as well as a UVW-certificate for each of these matrices which are non-zero). While verifying the validity of an LS_+^3 certificate package generally requires checking a much greater number of matrices and conditions compared to LS_+ and LS_+^2 certificate packages, each condition can still be checked reliably as, again, they only depend on elementary arithmetic operations on integers. Using these ideas and following a similar proof to that of Proposition 17, we have the following fact.

Proposition 28. *Let G be a graph. Then $r_+(G) \geq 4$ if and only if there exist a valid inequality $a^\top x \leq \beta$ for* $\text{STAB}(G)$ *and an* LS_+^3 *certificate package* ($Y, \mathcal{M}_1, \mathcal{M}_2$, and UVW-certificates) *for G such that* $\left(-\beta, a^\top\right) Y e_0 > 0$.

For this work, we performed a computational search for 4-minimal graphs, and found the following.

Theorem 29. *There are at least* 4107 *non-isomorphic* 4*-minimal graphs.*

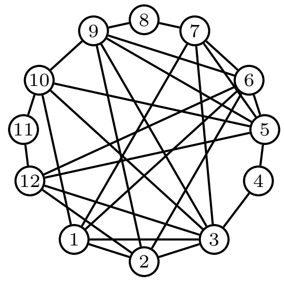
Proof. First, we are able to generate LS_+^3 certificate packages for 570 graphs G, showing that there exists $\bar{x} \in \text{LS}_+^3(G) \setminus \text{STAB}(G)$ for these graphs. Next, Lemma 15 implies that many edge subgraphs of these 570 graphs are also 4-minimal. Collecting these edge subgraphs and checking for isomorphisms among them resulted in a total of 4107 distinct 4-minimal graphs. □

The LS_+^3 certificate packages, as well as a full list of 4-minimal graphs we found, are available at [AT25a]. Our dataset also includes MATLAB code which can be used to verify all LS_+, LS_+^2, and LS_+^3 certificate packages mentioned in this manuscript. Furthermore, all matrices in our data are stored as widely-supported CSV (comma-separated values) files, so one can also verify and analyze these certificate packages in other programming languages, such as Python and R.

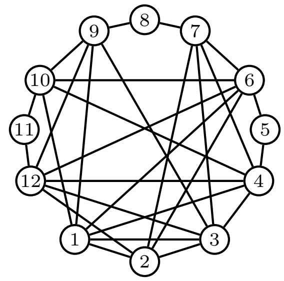
Next, we highlight a few of the 4-minimal graphs we found with notable features in Figure 14. For each graph we provide its encoding in graph6 format, the facet of its stable set polytope with LS_+-rank four, and the integrality ratio of this inequality according to CVX+SeDuMi. (The reader may refer to [McK] for a detailed description of graph6, an encoding of undirected graphs as strings of printable ASCII characters.)

Among the 4107 4-minimal graphs we found, 2318 belong to $\mathcal{K}_{6,3}$, including the aforementioned 588 graphs in $\hat{\mathcal{K}}_{6,3}$ with clique number at most three. Thus, in addition to the analytical proof from [AT25b], we now have an independent numerical proof that every graph $G \in \hat{\mathcal{K}}_{6,3}$ with $\omega(G) \leq 3$ is indeed 4-minimal. The densest 4-minimal stretched clique we found have 28 edges — $G_{14,1}$ and $G_{14,2}$ are two such examples.

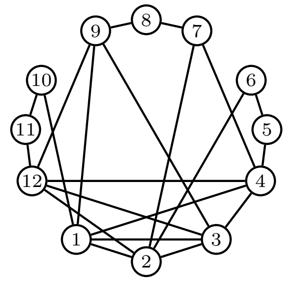
The densest 4-minimal graphs we discovered overall have 29 edges — $G_{14,4}$ and $G_{14,5}$ give two such instances. On the other hand, the sparsest 4-minimal graphs we found have 21 edges — there are 40 of them, all of which are sparse stretched cliques in $\hat{\mathcal{K}}_{6,3}$. One of these graphs is $G_{14,3}$, which has the notable feature that its facet of LS_+-rank four has the largest value of



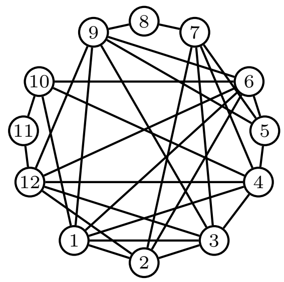
$G_{14,1} \coloneqq$ `KxFLWDliG?l``
$(1,1,1,1,1,1,1,1,1,1,1,1)^\top x \leq 4$
$\gamma_3(G_{14,1}, a) = 1.0008772$



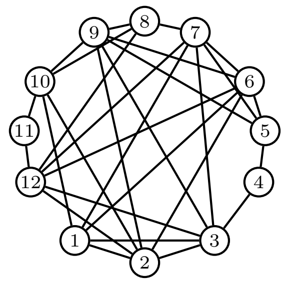
$G_{14,2} \coloneqq$ `K|FJgE`dG?md`
$(1,1,1,1,1,1,1,1,1,1,1,1)^\top x \leq 4$
$\gamma_3(G_{14,2}, a) = 1.0007099$



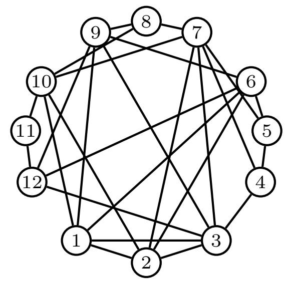
$G_{14,3} \coloneqq$ `K|DI_E`_??mD`
$(1,1,1,1,1,1,1,1,1,1,1,1)^\top x \leq 4$
$\gamma_3(G_{14,3}, a) = 1.0016810$



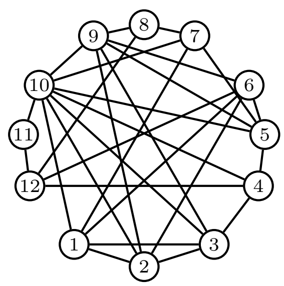
$G_{14,4} \coloneqq$ `K|FJoEld??md`
$(1,1,1,1,1,1,2,1,1,1,1,1)^\top x \leq 4$
$\gamma_3(G_{14,4}, a) = 1.0001388$



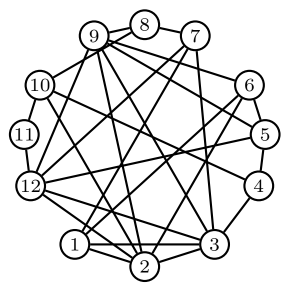
$G_{14,5} \coloneqq$ `KxFLWDloW?kx`
$(1,1,1,1,1,1,1,1,1,2,1,1)^\top x \leq 4$
$\gamma_3(G_{14,5}, a) = 1.0001323$



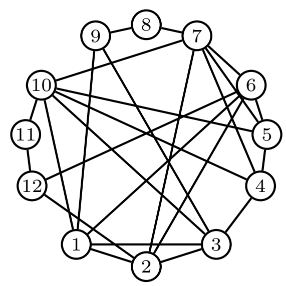
$G_{14,6} \coloneqq$ `KxFJoEdoo?cd`
$(1,1,1,1,1,1,2,1,1,2,1,1)^\top x \leq 4$
$\gamma_3(G_{14,6}, a) = 1.0000859$



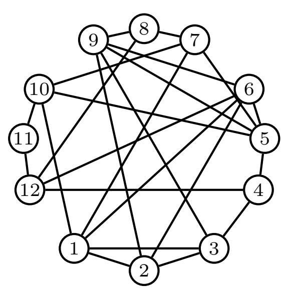
$G_{14,7} \coloneqq$ `KxFKODl}g?ah`
$(1,1,1,1,1,1,1,1,1,2,1,2)^\top x \leq 4$
$\gamma_3(G_{14,7}, a) = 1.0001203$



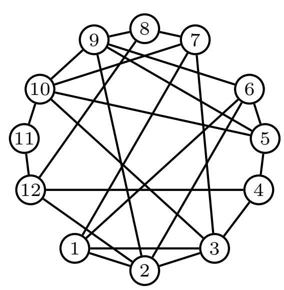
$G_{14,8} \coloneqq$ `KxFL?DlSO?lT`
$(1,1,1,1,1,1,1,1,1,2,1,1)^\top x \leq 4$
$\gamma_3(G_{14,8}, a) = 1.0003666$



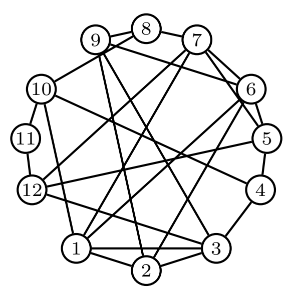
$G_{14,9} \coloneqq$ `KxFIwE`m_?g``
$(1,1,1,1,1,1,2,1,1,2,1,1)^\top x \leq 4$
$\gamma_3(G_{14,9}, a) = 1.0000789$



$G_{14,10} \coloneqq$ `KxFKODla_?ah`
$(1,1,1,1,1,1,1,1,1,1,1,2)^\top x \leq 4$
$\gamma_3(G_{14,10}, a) = 1.0003558$



$G_{14,11} \coloneqq$ `KxFL?DLIg?iH`
$(1,1,1,1,1,1,1,1,1,1,1,2)^\top x \leq 4$
$\gamma_3(G_{14,11}, a) = 1.0003679$



$G_{14,12} \coloneqq$ `KxFKWDdcO?dP`
$(1,1,1,1,1,1,1,1,1,2,1,1)^\top x \leq 4$
$\gamma_3(G_{14,12}, a) = 1.0002711$

FIGURE 14. A sample of twelve 4-minimal graphs with notable features

$\gamma_3(G,a)$ among all 12-vertex graphs we tested. The patterns we have seen from 3- and 4-minimal stretched cliques suggest the following.

Conjecture 30. For every positive integer ℓ, a sparsest ℓ-minimal graph is a sparse stretched clique in $\hat{\mathcal{K}}_{\ell+2,\ell-1}$.

Conjecture 31. For every positive integer ℓ, the maximum value of $\gamma_{\ell-1}(G,a)$ among ℓ-minimal graphs is attained by a sparse stretched clique $G \in \hat{\mathcal{K}}_{\ell+2,\ell-1}$ and $a := \bar{e}$.

Next, $G_{14,7}$, $G_{14,8}$, and $G_{14,9}$ give examples of 4-minimal graphs which contain K_4 as an induced subgraph. In particular, both $\{1,2,3,10\}$ and $\{2,3,9,10\}$ induce a K_4 in $G_{14,7}$, which is the only 4-minimal graph we found with multiple K_4's induced as subgraphs. Also, notice that these graphs do not belong to $\mathcal{K}_{6,3}$.

In other words, as with 3-minimal graphs, we did not find any 4-minimal graphs in $\mathcal{K}_{6,3}$ where $\omega(G) \geq 4$. In fact, one can exhaustively search and find that there are exactly 121 non-isomorphic sparse stretched cliques $G \in \hat{\mathcal{K}}_{6,3}$ where $\bar{e}^T x \leq 4$ is a facet of $\text{STAB}(G)$. 40 of them have $\omega(G) = 3$, all of which are among graphs which have been shown to be 4-minimal in Theorem 29. The remaining 81 all have $\omega(G) \geq 4$, and one can use similar ideas as in the proof of Proposition 26 to show that all 81 graphs have LS_+-rank at most 3. Thus, we can use the same argument for Proposition 26 to show that $\bar{e}^T x \leq 4$ has LS_+-rank at most 3 for every graph $G \in \mathcal{K}_{6,3}$ with $\omega(G) \geq 4$. This leads us to believe the following.

Conjecture 32. For every positive integer ℓ, if $G \in \mathcal{K}_{\ell+2,\ell-1}$ and $\omega(G) \geq 4$, then $r_+(G) \leq \ell - 1$.

If Conjecture 32 holds, then combining it with Theorem 7 gives that a graph $G \in \hat{\mathcal{K}}_{\ell+2,\ell-1}$, G is ℓ-minimal if and only if $\omega(G) \leq 3$.

Next, recall our discussion around Corollary 24 about 3-minimal graphs where the only way to obtain these graphs from K_3 using 1-join and 2-stretch operations is to alternate between these two operations. We found that there are also many such examples for 4-minimal graphs — $G_{14,6}$ and $G_{14,9}$ are two such instances.

Finally, recall that we mentioned earlier that every one of 49 3-minimal graphs in Figure 8 contains a stretched clique in $\mathcal{K}_{5,2}$ as an edge subgraph. This is no longer true for 4-minimal graphs, as $G_{14,10}$, $G_{14,11}$, and $G_{14,12}$ provide instances which do not contain a stretched clique in $\mathcal{K}_{6,3}$ as an edge subgraph. In particular, $G_{14,12}$ does not even contain K_6 as a graph minor, giving the first example of an ℓ-minimal graph which does not contain $K_{\ell+2}$ as a graph minor.

Next, we go into more detail about our approach for searching for 4-minimal graphs. First, we used the criteria described in Theorem 4 to construct a pool of viable candidates for being 4-minimal graphs. Let $\mathcal{X}_4$ denote the set of ordered pairs (G,a) with the following properties:

- $V(G) = [12]$, and the vertices in $[9] \subset V(G)$ induce one of the 49 3-minimal graphs listed in Figure 8.
- $\deg(11) = 2$, with $\Gamma_G(11) = \{10,12\}$. Also, $\Gamma_G(10)$ and $\Gamma_G(12)$ are not subsets of each other. (This is to ensure that G can be obtained from a proper 2-stretching of another graph.)
- a is the direction of a full-support facet of G using the criterion described in [AT24a, Corollary 15].

An exhaustive search found that $|\mathcal{X}_4| = 2038174$ after eliminating redundant isomorphic graphs. A straightforward approach would then be to compute $\gamma_3(G,a)$ for each of these elements, and then proceed to generate LS_+^3 certificate packages for graphs with integrality ratios comfortably above 1. However, optimizing over (a straightforward formulation of) $\text{LS}_+^3(G)$ with CVX+SeDuMi for a 12-vertex graph G takes 8-10 minutes per instance on our machine, which means that computing $\gamma_3(G,a)$ for all elements in $\mathcal{X}_4$ this way would take more than 30 years. Thus, we need additional insights to narrow down our search.

Therefore, let us focus on a particular subset of $\mathcal{X}_4$ by imposing the edge subgraph partial order on this set: Given $(G,a),(G',a') \in \mathcal{X}_4$, we define $(G,a) \leq (G',a')$ if $a = a'$ and G is an edge subgraph of G'. In this case, it follows from Lemma 15 that $r_+(G') = 4$ implies $r_+(G) = 4$. Thus, let $\bar{\mathcal{X}}_4$ to be the set of elements $(G,a) \in \mathcal{X}_4$ which are minimal with respect to this partial order. Then we know that every 4-minimal graph contains at least one graph in $\bar{\mathcal{X}}_4$ as an edge subgraph. Also, we have $|\bar{\mathcal{X}}_4| = 6822$, which is a much more manageable set to work with. Figure 15 shows the plot of the values of $\gamma_3(G,a)$ on the real number line for all $(G,a) \in \bar{\mathcal{X}}_4$.

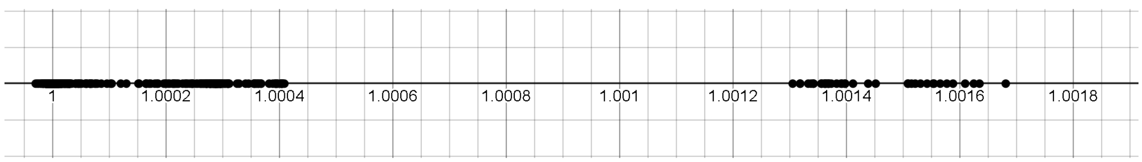


FIGURE 15. Plotting $\gamma_3(G,a)$ for all $(G,a) \in \bar{\mathcal{X}}_4$

Next, observe that for every $(G,a) \in \mathcal{X}_4$,

$$\gamma_3(G,a) \leq \max_{(G',a) \in \bar{\mathcal{X}}_4} \left\{ \gamma_3(G',a) : (G',a) \leq (G,a) \right\}. \tag{1}$$

Using (1), we obtain an upper bound on $\gamma_3(G,a)$ for every $(G,a) \in \mathcal{X}_4$. Then we selectively computed the actual value of $\gamma_3(G,a)$ for elements which are promising, resulting in the collection of 4-minimal graphs described in Theorem 29. With this reduction, we ended up computing $\gamma_3(G,a)$ for around 16000 graphs in $\mathcal{X}_4$. At 8-10 minutes per graph, solving these 16000 SDPs alone took CVX+SeDuMi more than 2000 hours of computation time.

Next, let us take a closer look at Figure 15. First, notice that there is a cluster of values above 1.001 — they consist of 40 elements in $\bar{\mathcal{X}}_4$, which are exactly the aforementioned 40 sparse stretched cliques in $\mathcal{K}_{6,3}$ with clique number at most 3. In particular, the rightmost dot in Figure 15 represents the integrality ratio of $G_{14,3}$.

In addition to these 40 sparse stretched cliques, there are another 134 graphs in $\bar{\mathcal{X}}_4$ which are certified to be 4-minimal in Theorem 29, with integrality ratios between 1.00004 and 1.0005. Unlike the case in Figure 12, there is not a clean visual break indicating which graphs "apparently" have integrality ratios above 1, and so we suspect that there could be more than 174 4-minimal graphs among elements of $\bar{\mathcal{X}}_4$.

Recall that in our analysis on $\mathcal{X}_3$, we looked into the possible predictive value of $\gamma_1(G,a)$ (computationally easier to obtain) for $\gamma_2(G,a)$ (actual indicator of whether a graph has LS_+-rank 3). Likewise, it is natural to wonder if $\gamma_1(G,a)$ and/or $\gamma_2(G,a)$ can serve as a heuristic in our search for 4-minimal graphs. In particular, between $\gamma_1(G,a)$ and $\gamma_2(G,a)$, which has better predictive value for $\gamma_3(G,a)$? To that end, we plotted $\gamma_1(G,a)$ against $\gamma_3(G,a)$ (Figure 16, left), as well as $\gamma_2(G,a)$ against $\gamma_3(G,a)$ (Figure 16, right). A simple linear regression shows that $\gamma_2(G,a)$ ($r \approx 0.3158$) indeed has a stronger correlation with $\gamma_3(G,a)$ compared to $\gamma_1(G,a)$ ($r \approx 0.03976$) for this particular set of graphs and facets. This is not surprising, as $\gamma_2(G,a)$ (3-4 seconds per instance) is more computationally costly to obtain than $\gamma_1(G,a)$ (roughly 0.25 seconds per instance) and should reveal more information about the underlying graph. In particular, as the clusters of points on the lower-left corner of both scatterplot suggest, graphs with the lowest $\gamma_1(G,a)$ and $\gamma_2(G,a)$ values all seem to have $\gamma_3(G,a)$ very close to 1.

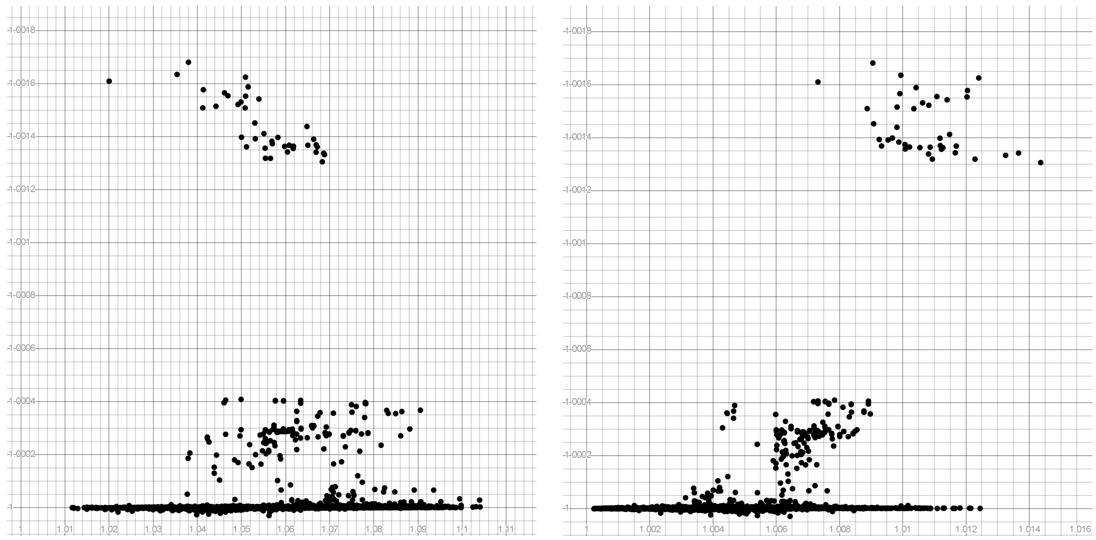


FIGURE 16. Scatterplots of $\gamma_1(G,a)$ versus $\gamma_3(G,a)$ (left) and $\gamma_2(G,a)$ versus $\gamma_3(G,a)$ (right) for all $(G,a) \in \bar{\mathcal{X}}_4$

5. Vertex-transitive graphs

In this section, we look into the smallest vertex-transitive graphs with a given LS_+-rank. As we saw, ℓ-minimal graphs must contain some (but not all) vertices with degree 2, making them necessarily irregular. Could restricting ourselves to highly-symmetric graphs expose new structures that make instances of the stable set problem challenging for LS_+?

Given $\ell \in \mathbb{N}$, let $\overline{n}_+(\ell)$ denote the minimum number of vertices among vertex-transitive graphs with LS_+-rank exactly ℓ. Theorem 2 implies that $\overline{n}_+(\ell) \geq 3\ell$ for every $\ell \in \mathbb{N}$, and the aforementioned result by Stephen and the second author [ST99] on the line graphs of odd cliques show that $\overline{n}_+(\ell) \leq \binom{2\ell+1}{2} = 2\ell^2 + \ell$ for every $\ell \in \mathbb{N}$. Both bounds are tight for $\ell = 1$, as $\overline{n}_+(1) = 3$ is attained by K_3.

Also, recall the graphs $\mathcal{B}_k$ defined in Section 1. Then Theorem 9 implies that

$$\min \{\overline{n}_+(k) : k \geq \ell\} \leq 4\ell + 8$$

for every odd $\ell \in \mathbb{N}$. Note that since we do not know the exact rank of $\mathcal{B}_k$ in general, Theorem 9 alone does not provide a bound for $\overline{n}_+(\ell)$ for a specific ℓ.

Next, given a graph G and $\ell \in \mathbb{N}$, define

$$\alpha_{\text{LS}_+^{\ell}}(G) := \max \left\{ \bar{e}^{\top} x : x \in \text{LS}_+^{\ell}(G) \right\}.$$

To show that $r_+(G) > \ell$, it is sufficient to show that $\alpha_{\text{LS}_+^{\ell}}(G) > \alpha(G)$. The following result will also be helpful.

Lemma 33. *Suppose G is a k-regular vertex-transitive graph with n vertices and $r_+(G) = \ell \geq 2$. Then $3 \leq k \leq n + 2 - 3\ell$.*

Proof. If $k \leq 2$, then each component in G is either a single vertex ($k = 0$), an edge ($k = 1$), or a cycle ($k = 2$), and $r_+(G) \leq 1$ in all of these cases. Thus, it follows that $k \geq 3$. Next, since G is vertex-transitive, $G \ominus i$ is isomorphic for all $i \in V(G)$, and so $r_+(G \ominus i) \geq \ell - 1$ (due to Theorem 14), and thus $|V(G \ominus i)| \geq 3\ell - 3$. Since the destruction of i removes $k+1$ vertices from G, this shows that $n \geq 3\ell + k - 2$, and the claim follows. $\square$

A well-studied family of vertex-transitive graphs which we will frequently refer to is the circulant graphs. Given $S \subseteq [n]$, we define the *circulant graph* C_n^S where $V(C_n^S) := [n]$ and

$$E(C_n^S) := \{\{i,j\} : \ (j-i) \bmod n \in S \text{ or } (i-j) \bmod n \in S\}.$$

Next, we will show that the graphs $G_{17,1}$, $G_{17,2}$, and $G_{17,3}$ from Figure 17 are the smallest vertex-transitive graphs with LS_+-rank 2, 3, and 4, respectively. This shows that $\overline{n}_+(2) = 8$, $\overline{n}_+(3) = 13$, and $\overline{n}_+(4) = 16$. Notice that $G_{17,1} = C_8^{\{1,2\}}$ and $G_{17,2} = C_{13}^{\{1,5\}}$ are both circulant graphs. For $G_{17,3}$, one way to make sense of the graph is to observe that it consists of a copy of $C_8^{\{1,4\}}$ (induced by vertices $\{1,\ldots,8\}$) and a copy of $C_8^{\{3,4\}}$ (induced by vertices $\{9,\ldots,16\}$) joined together by a perfect matching.

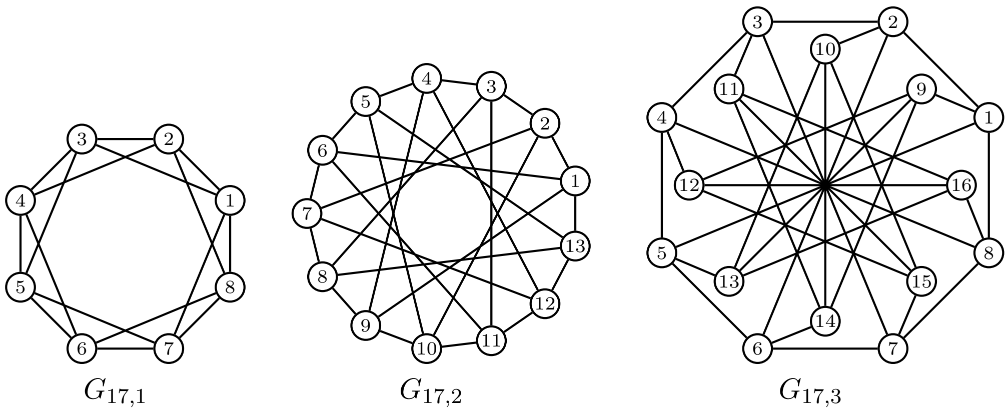


FIGURE 17. The smallest vertex-transitive graphs with LS_+-rank 2 (left), 3 (centre), and 4 (right)

A significant portion of our work in this section involves proving LS_+-rank upper bounds for vertex-transitive graphs up to a certain size, which is largely aided by the fact that vertex-transitive graphs on up to 47 vertices have been exhaustively listed (see [MR90, Ski, RH20]).

We now determine $\overline{n}_+(2)$.

Proposition 34. $\overline{n}_+(2) = 8$. *Moreover, the only vertex-transitive graph G where $|V(G)| \leq 8$ and $r_+(G) = 2$ is $G_{17,1} := C_8^{\{1,2\}}$ (Figure 17, left).*

Proof. Suppose G is a k-regular vertex-transitive graph on n vertices, and $r_+(G) = 2$. To find the smallest such graph, we may assume that G is connected (otherwise G contains a proper subgraph with the same LS_+-rank). From Lemma 33, we know that $3 \leq k \leq n-4$, and so $n \geq 7$. If $n = 7$, then $k = 3$, and no such graph exists. If $n = 8$, then $3 \leq k \leq 4$. There are a total of 5 such graphs [MR90, Ski], as listed in Figure 18.

Observe that $G_1 = C_8^{\{1,4\}}$, and destroying any vertex yields a bipartite graph, and so $r_+(G_1) = 1$. Next, G_2 (the 3-cube) and G_3 ($K_{4,4}$) are both bipartite and thus have LS_+-rank 0. G_4 is obtained from joining two disjoint copies of K_4 with a perfect matching, and thus is a perfect graph, which implies that $r_+(G_4) = 1$. This leaves us with G_5, which is the graph $G := G_{17,1}$ from Figure 17. There are several ways to show that $r_+(G) \geq 2$:

- An LS_+ certificate package [AT25a] shows that $\frac{4}{15}\bar{e} \in \text{LS}_+(G)$, which implies that $\alpha_{\text{LS}_+}(G) \geq \frac{32}{15} > \alpha(G) = 2$. Thus, $r_+(G) \geq 2$.

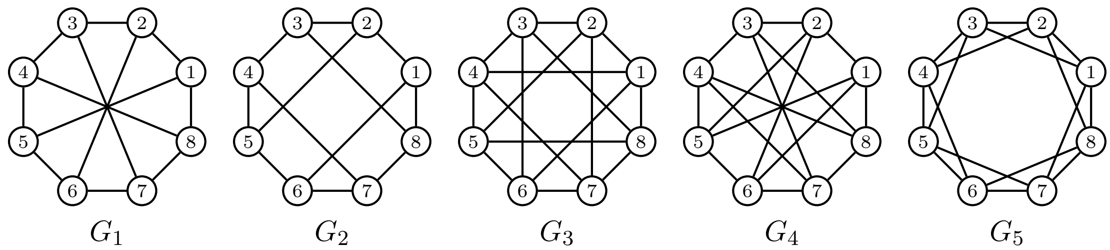


FIGURE 18. The list of connected k-regular vertex-transitive graphs on 8 vertices with $3 \leq k \leq 4$

- Let $A(\overline{G})$ be the adjacency matrix of the complement graph of G, let $d := \frac{\sqrt{2}+4}{8(\sqrt{2}+1)}$, and let $Y := \begin{bmatrix} 1 & d\bar{e}^\top \\ d\bar{e} & dI_8 + \frac{d}{\sqrt{2}+1}A(\overline{G}) \end{bmatrix}$. Then one can check that $Y \in \widehat{\mathrm{LS}}_+(G)$ (see, for instance, [ALT22, Proposition 4] for a more detailed analysis for the LS_+ certificates for regular graphs and some vertex-transitive graphs). This shows that $\alpha_{\mathrm{LS}_+}(G) \geq 8d \approx 2.24 > \alpha(G)$.
- Observe that $G - \{1, 3\}$ is isomorphic to $G_{1,2}$. Since G contains an induced subgraph with LS_+-rank 2, $r_+(G) \geq 2$.

Finally, since $|V(G)| = 8, r_+(G) \leq 2$ (by Theorem 2). It thus follows that $r_+(G) = 2$. $\square$

We remark that, for $n = 9$, there is also exactly one vertex-transitive graph with LS_+-rank 2: $C_9^{\{1,2\}}$ (which contains $G_{1,1}$ as an induced subgraph). The other two graphs which satisfy the criterion in Lemma 33 are $C_9^{\{1,3\}}$ and the Paley graph on 9 vertices, both of which would result in a bipartite graph upon the destruction of any vertex.

Next, we remark that while $\alpha_{\mathrm{LS}_+^\ell}(G) > \alpha(G)$ implies $r_+(G) > \ell$, the converse is not true. Before we describe such an example, we first prove a more general result.

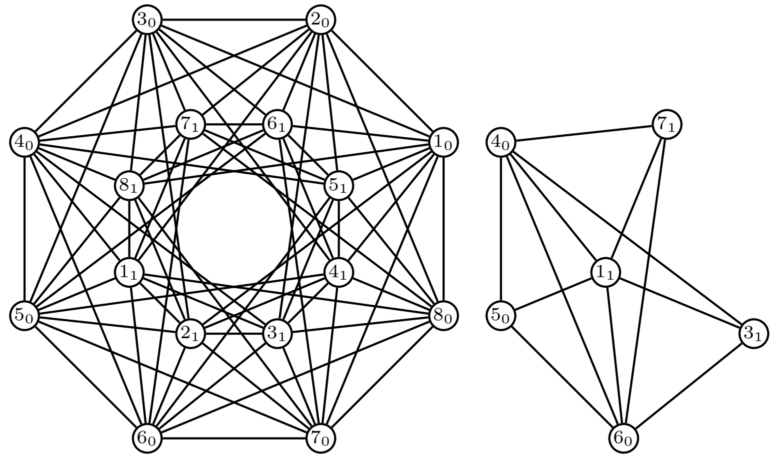
Proposition 35. *Suppose G is a graph with n vertices and* $\deg(i) \geq k$ *for every vertex $i \in V(G)$. Then*

(i)
$$\alpha_{\mathrm{LS}_+}(G) \leq n - k.$$

(ii) *Moreover, if* $\max\left\{\bar{e}^\top x : x \in \mathrm{FRAC}(G \ominus i)\right\} \leq \frac{1}{2}|V(G \ominus i)|$ *for every vertex $i \in V(G)$, then*
$$\alpha_{\mathrm{LS}_+}(G) \leq \frac{n-k+1}{2}.$$

Proof. Let $Y := \begin{bmatrix} 1 & x^\top \\ x & \mathrm{Diag}(x) + M \end{bmatrix}$ be a certificate matrix in $\widehat{\mathrm{LS}}_+(G)$. Then $Y \succeq 0$, and thus the Schur complement $\mathrm{Diag}(x) + M - xx^\top$ is also positive semidefinite. Hence,
$$\bar{e}^\top(\mathrm{Diag}(x) + M - xx^\top)\bar{e} = \bar{e}^\top x - (\bar{e}^\top x)^2 + \bar{e}^\top M\bar{e} \geq 0.$$
Next, notice that since $Ye_0 = \mathrm{diag}(Y)$, $M[i,i] = 0$ for every $i \in V(G)$. Also, $M[i,j] = 0$ for all edges $\{i,j\} \in E(G)$. Thus, for every $i \in V(G)$, Me_i must contain at most $n - \deg(i) - 1 \leq n - k - 1$ positive entries, each being at most x_i. Applying this for every $i \in V(G)$ yields $\bar{e}^\top M\bar{e} \leq (n-k-1)\bar{e}^\top x$. This gives $\bar{e}^\top x \leq n - k$.

For (ii), the additional assumption assures that $Me_i \leq \frac{n-k-1}{2}x_i$, and so $\bar{e}^\top M\bar{e} \leq \frac{n-k-1}{2}\bar{e}^\top x$, leading to the tighter bound $\bar{e}^\top x \leq \frac{n-k+1}{2}$. $\square$



G $\quad\quad$ $G \ominus 1_0$

FIGURE 19. A vertex-transitive graph G where $\alpha_{\mathrm{LS}_+}(G) = \alpha(G)$ and $r_+(G) > 1$

Example 36. Consider the graph G where $V(G) \coloneqq \{i_0, i_1 : i \in [8]\}$ and

$$E(G) \coloneqq \left\{ \{i_0, j_0\}, \{i_1, j_1\} : \{i,j\} \in E(C_8^{\{1,2\}}) \right\} \cup \left\{ \{i_0, j_1\} : \{i,j\} \in E(C_8^{\{1,3,4\}}) \right\}.$$

(See Figure 19, left.) Then G is $k \coloneqq 9$-regular and vertex-transitive, with $G \ominus 1_0$ being a 6-vertex graph (Figure 19, right). Now notice that the edges $\{4_0, 7_1\}$, $\{5_0, 1_1\}$, and $\{6_0, 3_1\}$ form a perfect matching in $G \ominus 1_0$. Thus, $\max \left\{ \bar{e}^\top x : x \in \mathrm{FRAC}(G \ominus 1_0) \right\} \leq 3 = \frac{1}{2}|V(G \ominus 1_0)|$. Since G is vertex-transitive, the same must be true for all vertices $i \in V(G)$, and so G satisfies the conditions of Proposition 35(ii). Hence, $\alpha_{\mathrm{LS}_+}(G) \leq \frac{16-9+1}{2} = 4 = \alpha(G)$. However, since G contains $C_8^{\{1,2\}}$ as an induced subgraph, we see that $r_+(G) \geq 2$.

We next determine $\overline{n}_+(3)$.

Proposition 37. $\overline{n}_+(3) = 13$. *Moreover, the only vertex-transitive graph G where $|V(G)| \leq 13$ and $r_+(G) = 3$ is $G_{17,2} \coloneqq C_{13}^{\{1,5\}}$ (Figure 17, centre).*

Proof. Suppose a vertex-transitive graph G has n vertices, is k-regular and has LS_+-rank 3. It follows from Lemma 33 that $3 \leq k \leq n-7$. Thus, we see that $n \geq 10$. As with in the proof of Proposition 34, we may assume that G is connected.

There are 38 connected vertex-transitive graphs on $n \in \{10, 11, 12, 13\}$ vertices which satisfy the degree bounds from Lemma 33 with $\ell = 3$ [RH20], which we list in Table 1. (For compactness, we are listing these graphs in graph6 format.)

Next, notice that

- For every $i \in \{1, 6, 8, 10, 11, 12, 20\}$, G_i is bipartite, so these graphs all have LS_+-rank 0.
- For every $i \in \{2, 3, 4, 7, 21, 26, 32, 35\}$, $G_i - 1$ is bipartite, so these graphs all have LS_+-rank 1.
- For every $i \in \{5, 9, 14, 15, 17, 18, 19, 22, 23, 24, 25, 27, 28, 29, 30, 34, 36, 37, 38\}$, $G_i \ominus 1$ is perfect, so these graphs all have LS_+-rank at most 2.
- Notice that $(G_{13} \ominus 1) - 12$ and $(G_{16} \ominus 1) - 8$ are both bipartite, and so $r_+(G_{13}), r_+(G_{16}) \leq 2$.
- Finally, notice that $G_{31} \ominus 1$ is the 5-wheel, which has LS_+-rank 1. Thus, $r_+(G_{31}) \leq 2$.

i	G_i	i	G_i	i	G_i	i	G_i
1	Is?@WxcU?	11	Ks_?BLUR`wFO	21	KsaIPKuUZkNG	31	K}iWBDRIol@r
2	Is?HGtcU?	12	Ks_?BLeF_{N?	22	KseXa`JHrJLQ	32	Ls_?GSTTPTLO\?
3	IsP@OkWHG	13	Ks_GagjLASko	23	KseY`TIKZLKi	33	Ls`?XGRQR@B`Kc
4	Js`@IStU`w?	14	Ksc?JLSQhSE`	24	KsiYISiDZdMI	34	Lts?GKEPPDHIKI
5	Juk?IKeTPT?	15	Ksc@ILKLAdDI	25	KsiZ?lEEZBnO	35	L|qC@|]JakiiIj
6	Ks???wYP`KL?	16	Ksd@?STWY[Eo	26	KtaHGthYaiis	36	L}akqXXWomdULJ
7	Ks?GOGUIQcKG	17	Ktk?IHBD_[kK	27	KtiWBDRQqLfo	37	L}nDAwyBgmGfGv
8	Ks?GOObDRCI_	18	Kuk?GLDGqhDQ	28	KtiY@DFQYefo	38	L~zTQgiDOT_n?~
9	Kt?GOHAOWsCg	19	K{cAGgeBQSeK	29	K{eY`dIPhRCj		
10	Ks_?BLMLasF_	20	Ksa?BtuZa{Fo	30	K{fw?DbWwuBX		

TABLE 1. List of connected vertex-transitive graphs satisfying the degree bounds from Lemma 33 with $n \in \{10, 11, 12, 13\}$ and $\ell = 3$, in graph6 format

This leaves G_{33}, which is isomorphic to $G \coloneqq C_{13}^{\{1,5\}}$. We provide a LS_+^2 certificate package [AT25a] showing that $\frac{17}{55}\bar{e} \in LS_+^2(G)$, which implies that $\alpha_{LS_+^2}(G) \geq \frac{221}{55} > \alpha(G) = 4$. This implies that $r_+(G) \geq 3$. Also, destroying a vertex in G yields an 8-vertex graph, which has LS_+-rank at most two by Theorem 2. Thus, it follows that $r_+(G) = 3$. $\square$

We next determine $\overline{n}_+(4)$ using a similar analysis.

Proposition 38. $\overline{n}_+(4) = 16$. *Moreover, the only vertex-transitive graph G where $|V(G)| \leq 16$ and $r_+(G) = 4$ is $G_{17,3}$ (Figure 17, right).*

Proof. Let G be a k-regular vertex-transitive graph on n vertices where $r_+(G) = 4$. Again, we may assume that G is connected. By Lemma 33, we know that $3 \leq k \leq n - 10$, so $n \geq 13$. When $n = 13$, $k = 3$, and no such graph exists. So we may assume $n \geq 14$.

Next, we look into the connected k-regular vertex-transitive graphs on $n \in \{14, 15, 16\}$ vertices which satisfy $3 \leq k \leq n - 10$. There are 96 such graphs [RH20], which are listed in Table 2.

Observe that

- For every $i \in \{1, 2, 4, 5, 16, 18, 19, 20, 21, 22, 26, 29, 33, 34, 35, 36, 58, 59, 60\}$, G_i is bipartite, so these graphs all have LS_+-rank 0.
- For every $i \in \{3, 6, 11, 17, 37, 38, 61\}$, $G_i - 1$ is bipartite, so these graphs all have LS_+-rank 1.
- For every $i \in \{8, 12, 13, 15, 31, 32, 42, 45, 46, 53, 57, 65, 66, 67, 69, 73, 74, 75, 76, 77, 78, 79, 80, 81, 83, 84, 86, 88, 89, 90, 92, 96\}$, $G_i \ominus 1$ is perfect, so these graphs all have LS_+-rank at most 2.
- The graphs

$$\begin{array}{llll} (G_7 \ominus 1) - 6, & (G_9 \ominus 1) - 7, & (G_{10} \ominus 1) - 7, & (G_{23} \ominus 1) - 14, \\ (G_{24} \ominus 1) - 14, & (G_{25} \ominus 1) - 6, & (G_{27} \ominus 1) - 6, & (G_{28} \ominus 1) - 7, \\ (G_{39} \ominus 1) - 7 & & & \end{array}$$

  are all bipartite. Thus, $r_+(G_i) \leq 2$ for every $i \in \{7, 9, 10, 23, 24, 25, 27, 28, 39\}$.
- The graphs

$$(G_{14} \ominus 1) - \{6, 7\}, (G_{40} \ominus 1) - \{7, 8\}, (G_{52} \ominus 1) - \{7, 8\}$$

  are all bipartite. Thus, $r_+(G_i) \leq 3$ for every $i \in \{14, 40, 52\}$.

i	G_i	i	G_i	i	G_i
1	Ms???@KOpSBGHOD_?	33	Osa????Dw^KwXgUWFKBs?	65	OsaKYPDHOiDFEM[wMPRcg
2	Ms??OGKB?ccKS_WO?	34	Osa????ExN@{UWXgNGBk?	66	OsaKg?dQGid\[S[qLSQy_
3	Ms?GGSG@?bCQSGWC?	35	Osa????WyfH[PwN_Fg@x?	67	OsaKg?hSZEkkTIIdJWPyA
4	Ms_???VBpeHg[_Z??	36	Osa????WyuFKRWJgIs@m?	68	OsaKiCaC‘RbMYKTYLabiC
5	Ms_??@eTPeHWJOF_?	37	Osa??CH@gqctUSXgNGBk?	69	OsaKiCdPPDaUBR]WNAboS
6	Ms_??KEXbKBKEW]??	38	Osa??cQAWJIRKeZ_LgBX?	70	OsaSWSTOhDKiXQUEfBbr?
7	Ms_AHGHCihDaQOK__	39	Osa?AKUB‘KKEWKERBHRs?	71	OsaSXCdOha‘T[DRRNEAyC
8	Mts?GKE@QDCIQIKD?	40	Osa?OGgPGqasKsSwLGbKO	72	OsaSXDCSXRbKTBIdmSBY@
9	Ns_?@DDDOTDASBL_Ho?	41	OsaBA‘GP@‘dIHWEcas_]O	73	OsaSYHBGpH‘XDL]SNDBoK
10	Ns_?ACKChIIDISR_Eo?	42	OseW?CBSaPgjK_IGcgZF?	74	Osedw?DOYFHBSZW\Fg@w‘
11	Ns_?GGAAohdWTGYOMC?	43	OseW?CFOQ‘eAHPI‘h[BCo	75	Osedw?HOYFGbSZW\Fg@w‘
12	Nsc?GCCGyJI_I_QHEGO	44	OseW?CFOQ‘eAH‘IPh[BCo	76	Osedw@??pJhMP\UWEjBpA
13	Nsc?GCDP‘Bi_Q‘IOECG	45	OseW?CbOyHCaSaQQdO‘aH	77	OsfDw?@G‘bgmW\RWFFAyA
14	Nto?AHBHPC‘PCgD?‘_g	46	OseW?CbOyPCaSaKPGhPSI	78	OsfDw@@GXPCZP]TSFDayA
15	Nuk?GGA@oLCIOIIPICO	47	OseW?SFGHAiOKeQBGajF?	79	OsfLg?@WYHcZQYKXJcPuA
16	Os???GA?WBBAEAP_CoBO?	48	OseWOLAOYECXSLKADAHHB	80	Osqsw@@AXCclW]USEXaxA
17	Os?GOGEAOC?GCDGII?b?O	49	Oseg?D@AqcgfWwEOEOHHB	81	Osqsy@@GWRcdGtUEESqyA
18	Os?GOO??AHGQGSDCAK@D?	50	Osf_?KFO‘CBRSSKQdDAP‘	82	OtaCXOTHOphhRKSsKTJcK
19	Os?GOOD@_C?GCEGHI@B?G	51	Osf_GK_O@Ba[SLQcBJ@TA	83	OtaLw?@OaRgn[S[Kdk?z@
20	Os_????@zKIgIgLGHW?r?	52	OsiW??‘CjCiKPPQad[?m_	84	OtaLw?@ObBiNQ[P[fg@x@
21	Os_????AwjKgJ_RGESAe?	53	OsiW?CAOWNKQEQDWkUBoG	85	OtaLw?BOBBhNP[S[fa@y@
22	Os_????BWfKoTOLGHS@U?	54	OsiW?CBWieHEPRK_DCGiB	86	OtaLy@@AWJg[WFSFfg@x@
23	Os_??CDAGaiaIaX_HS@i?	55	OsiWA?RG_KiPWWQRAePU_	87	OtaLyD‘SYQGhKBKB‘eOXb
24	Os_??CDCGQePJCX_ISAY?	56	OsiWA@@AgMkXSkK_DCGiB	88	Ota\W@@GYChJS]PZFc@u@
25	Os_??DAGOZAqIGSCho@i?	57	O{fw?CB?wE?XWKWKb@_oX	89	Otakw?@QYbKLPLOufc@u@
26	Os_??KE@_KKKWWEEBBBo?	58	OsaC???FzrMkYwXwJw@|?	90	Otaky@@GWbhBPROlfc@u@
27	Os_??L@GOS‘SEKT@E‘BK?	59	OsaC???RxnNKYw\WJs@}?	91	OtrTOGBW@‘hIP]G|Bg_tP
28	Os_?A?_DOhA[BWQaDQBK?	60	OsaC???]ZrK{XwFwB{B{?	92	Ouj\w?@?YBcMSUILIhPTI
29	Os_?GGC@?RaUSgISLGBc?	61	OsaCB@_EWrKrXeFwB{B{?	93	O{fL_@HKQJCZCsBW_pzp_
30	Os_?IGbC‘Ga_O‘ISB@QPG	62	OsaKYCQGbRIjXKLWJEHqc	94	O{fL_CCQP‘GmCzB[C\Joo
31	Otk?GGB?WBgAOBD_@oJ?K	63	OsaKYCcQXRBKSbLDmTBi_	95	O}akqPPWOV@iHIDHcROcj
32	Ouk?GKC?_J?YGIODDCQ_a	64	OsaKYDDGgdLBUELQeibr?	96	O~aKYPDOxQBHHIGeacocj

Table 2. List of connected vertex-transtive graphs satisfying the degree bounds from Lemma 33 with $n \in \{14, 15, 16\}$ and $\ell = 4$, in graph6 format

- The graphs

$$\begin{array}{llll} (G_{43} \ominus 1) - 7, & (G_{44} \ominus 1) - 7, & (G_{47} \ominus 1) - 7, & (G_{48} \ominus 1) - 7, \\ (G_{49} \ominus 1) - 7, & (G_{54} \ominus 1) - 7, & (G_{55} \ominus 1) - 8, & (G_{56} \ominus 1) - 7, \\ (G_{63} \ominus 1) - 10, & (G_{64} \ominus 1) - 8, & (G_{68} \ominus 1) - 8, & (G_{70} \ominus 1) - 8, \\ (G_{71} \ominus 1) - 8, & (G_{72} \ominus 1) - 8, & (G_{85} \ominus 1) - 8, & (G_{87} \ominus 1) - 8, \\ (G_{93} \ominus 1) - 9, & (G_{94} \ominus 1) - 10 & & \end{array}$$

  are all perfect. Thus, $r_+(G_i) \leq 3$ for every $i \in \{43, 44, 47, 48, 49, 54, 55, 56, 63, 64, 68, 70, 71, 72, 85, 87, 93, 94\}$.
- For every $i \in \{62, 91, 95\}$, observe that $G_i \ominus 1$ has exactly 9 vertices, all of which have degree at least 3, and thus $G_i \ominus 1$ is not 3-minimal (since every 3-minimal graph must have at least one vertex with degree 2 due to Theorem 4). Hence $r_+(G_i \ominus 1) \leq 2$, which implies that $r_+(G_i) \leq 3$.
- Next, for convenience, let $H \coloneqq G_{50} \ominus 1$ (Figure 20, left). Observe that, for every $i \in V(H)$, $H \ominus i$ is either a 5-vertex graph, or a 6-vertex graph which is not isomorphic to $G_{1,1}$ or $G_{1,2}$. Thus, $r_+(H \ominus i) \leq 1$ for every $i \in V(H)$, which implies that $r_+(H) \leq 2$, and thus $r_+(G_{50}) \leq 3$. The same argument applies for $H \coloneqq G_{82} \ominus 1$ (Figure 20, centre). Thus, $r_+(G_{50}), r_+(G_{82}) \leq 3$.
- Next, consider the graph $H \coloneqq (G_{51} \ominus 1) - 10$ (Figure 20, right). Notice that $\{8, 11\}$, $\{9, 12\}$, and $\{7, 16\}$ are all cut cliques in H, and one can show that $r_+(H) \leq 1$ by decomposing H using Proposition 13. Thus implies that $r_+(G_{51}) \leq 3$.
- Finally, G_{41} is the (5-regular) Clebsch graph, and so $G_{41} \ominus 1$ yields the Peterson graph, which has LS$_+$-rank 1. (To see this, observe that the Peterson graph is vertex-transitive in its own right, and destroying any vertex yields a bipartite graph.) Thus, $r_+(G_{41}) \leq 2$.

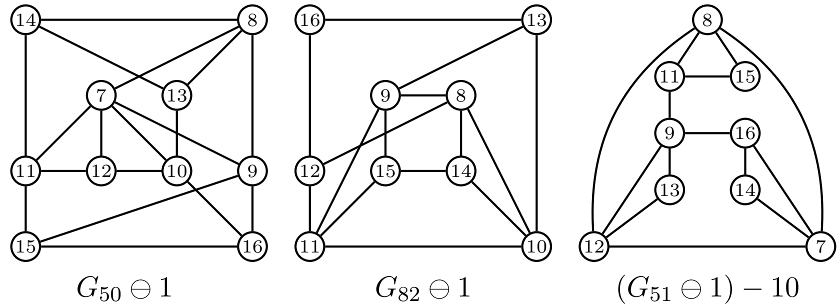


FIGURE 20. Illustrations for the proof of Proposition 38

This leaves G_{30}, which is isomorphic to $G \coloneqq G_{17,3}$ in Figure 17. The attached LS_+^3 certificate package [AT25a] shows that $\frac{26}{81}\bar{e} \in LS_+^3(G)$, and thus $\alpha_{LS_+^3}(G) \geq \frac{416}{81} > \alpha(G) = 5$, implying that $r_+(G) \geq 4$. Also, observe that $(G \ominus 1) - \{11, 15\}$ is a 9-cycle (LS_+-rank 1), implying that $r_+(G) \leq 4$. □

Coincidentally (or not?), $G_{17,1}$, $G_{17,2}$, and $G_{17,3}$ are all 4-regular. Also, all three graphs have some stretched-clique structures embedded in them, which are highlighted in Figure 21. First, as mentioned in the proof of Proposition 34, $G_{17,1} - \{1, 3\}$ gives a copy of $G_{1,2} \in \hat{\mathcal{K}}_{4,1}$. Also, notice that $G_{17,2} \ominus 1 \in \hat{\mathcal{K}}_{4,2}$ and $G_{17,3} \ominus 1 \in \hat{\mathcal{K}}_{5,3}$. Moreover, in both cases, one can join a new vertex to every vertex in the aforementioned stretched-clique subgraph and then 4-stretch the new vertex to obtain the full graph. Thus, these graphs seem to share some structural similarities with ℓ-minimal graphs, as they can also be obtained from K_3 by a sequence of 1-joining and k-stretching operations akin to that described in Corollary 24.

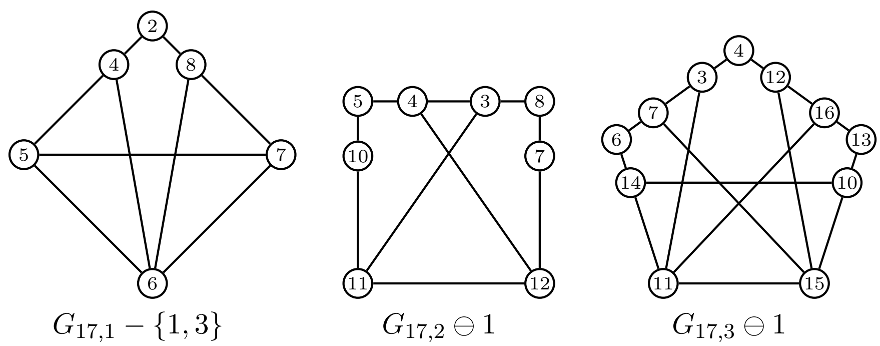


FIGURE 21. Notable stretch-clique induced subgraphs of $G_{17,1}$, $G_{17,2}$, and $G_{17,3}$

We end this section by proving a lower bound on $\overline{n}_+(5)$.

Proposition 39. $\overline{n}_+(5) \geq 20$.

Proof. Again, we may focus on connected vertex-transitive graphs which satisfy the degree bounds from Lemma 33. It follows from the proof of Proposition 38 that no 16-vertex graph in our consideration has LS_+-rank 5, so we may assume $n \geq 17$ here. There is a total of 58 such graphs on $n \in \{17, 18, 19\}$ vertices [RH20], as listed in Table 3. To prove our claim, it suffices to show that all of these graphs have LS_+-rank at most 4.

Observe that

- For every $i \in \{5, 6, 7, 10, 11, 12, 13, 15, 22, 23, 24, 25, 26\}$, G_i is bipartite, so these graphs all have LS_+-rank 0.

i	G_i	i	G_i	i	G_i
1	Ps_?GCD@?C@UDQIKIHBO_[A?	21	Qts?GKE@OD?I?I?DC@Q?IQ?SoA_	41	QtaIA@@O`E@T@UCLdA@H@EOGU?W
2	Ps_?GCDP?acOAHKcK_OqGQI?	22	Qsa??????V_}E[HkJIBL?]O@u??	42	QtiW?CB??B`T@rOKGIJ?`SCS}??
3	Ps_?GGAP@C@MCYRGEW@O_SAC	23	Qsa??????mb[IkP[Hs@Y_[c@p_?	43	QtiW?CB?GB?tArOSGBI_`WCS}??
4	Pts?GKE@OD?I?IGDG@QODK@O	24	Qsa?????@ed[D[BkKwBH_US@Y_?	44	Q{fw?CB?wE?X?K?Kk@b?XE?oWBG
5	Qs??OGC@?O@?GgCcCG_aOOD@?g?	25	Qsa?????AR`yHkD[MQ@r?Ug@f??	45	Rs_??CD@_KGAGAAhAgQP_KQ?gCA_CG
6	Qs??WOC@?O??O`GHAGaA_CI?_W?	26	Qsa?????BEkUW[EwA{?^?VC?{O?	46	Rs_?GCD@?C?S@OAKaH`OcSB@gOB_O?
7	Qs??WWG@?@?A?W?cA?h?SS@@_G?	27	Qsa??CA?OJ@RDXIeIkBS_\G@wO?	47	Rs_?GGA?O@CRPEAOACHU?TG@GG`_a?
8	Qs?GGSG@?@?A?W?cA?h?SS@@_G?	28	Qsa??CCAATCjDSEgGiHTC[W@qO?	48	Rts?GKE@OD?A?A?T?DP?DO@PGCH_C_
9	Qt?G?CG@?A_S?E?HG?I?DF??b??	29	Qsa??KCOObCrBAEcBUBDCWGpx??	49	RsaC??@?gI`TDTTTDTRTO\S@y_By??
10	Qs_?????@E`[C[BKIQ@k?RO@c_?	30	QsaG??A?QUGfHKDKDWOuA[g@pO?	50	RsaC?GAHAC`dCtLIHYHrOU[@kaBeG?
11	Qs_?????AHgUO[HoAw?]?MG?wO?	31	QsaG??A?XfIYHGQCak?s`XOPiA?	51	RsaKg??AGagiGiQZEMQZ_Ky@iCBgCG
12	Qs_?????AJCqA[CkIW@U?RC?s_?	32	QsaG??aCQdKIDHIEGkAS`HoOZA?	52	RsaKg??GYBg[HWSLDHYggIa`Y_Py?O
13	Qs_?????AK`MGsDKKgAd?Ig?\??	33	QsaH?_`CaCgKQoPKch?kGIDGWgO	53	RsaKg?@P@CDMSYCs@dJEcXJ@Wp@qP?
14	Qs_??GB@`@gIDAA`EcAU?I?`O@G	34	Qseg??B?oE_uOoSAceAaSIHHw@?	54	RsaSW_GGqHgXHKK@h?wj@AwPogJQ__
15	Qs_??KE@?C?oALCRDEAaOYC@oG?	35	QsiW??A?PDaMSgWSGoQQBDd?hg?	55	RsfLg?@?GAGNG]PTCTQPaKQGyGIw__
16	Qs_??KE@_K?K?WWEKB?oKE@`w??	36	QsiW??B?x?aDWSSgHGQKBIR?UK?	56	Rtq}w?@?WB_M?YOTCDQOiKAhIDPgDO
17	Qs_?G?C?aFCiAGCCdKAa_Q_OsA?	37	QsiW?D?OhCAIAW@qkAabCDBGig?	57	RujL_?@COP_[@WSTDDOdwBU`aDBGDG
18	Qsc???A?QCcKPgHS@a?h@I`@PG?	38	QsnO??ACWSGHOjGOCMGl_IaPDD?	58	R}akq?D?wOC@CBSQkI_ZQAfGce?dcG
19	Qsc?GGB@?C_oOXGFIGAP@IA?oOG	39	QtaG?@@OgU_{W`SQdA?h?HOKY?o		
20	Qsc?GKC@?D?IGYOdHA@`?IADOC_	40	QtaG?C@?jEIFQcPWdO@I@ECGS_W		

TABLE 3. List of connected vertex-transitive graphs on $n \in \{17, 18, 19\}$ vertices satisfying the degree bounds from Lemma 33 with $\ell = 5$

- For every $i \in \{1, 8, 16, 27, 46, 49\}$, $G_i - 1$ is bipartite, so these graphs all have LS_+-rank 1.
- For every $i \in \{4, 9, 19, 20, 21, 30, 34, 42, 43, 44, 48, 52, 55, 56\}$, $G_i \ominus 1$ is perfect, so these graphs all have LS_+-rank at most 2.
- The graph $(G_{17} \ominus 1) - 8$ is bipartite. Thus, $r_+(G_{17}) \leq 2$.
- The graphs

$$
\begin{array}{lll}
(G_2 \ominus 1) - \{6, 8\}\,, & (G_3 \ominus 1) - \{8, 9\}\,, & (G_{28} \ominus 1) - \{7, 9\}\,, \\
(G_{29} \ominus 1) - \{11, 14\}\,, & (G_{31} \ominus 1) - \{7, 12\}\,, & (G_{32} \ominus 1) - \{7, 8\}\,, \\
(G_{33} \ominus 1) - \{7, 9\}\,, & (G_{45} \ominus 1) - \{10, 11\}\,, & (G_{47} \ominus 1) - \{11, 12\}\,, \\
(G_{50} \ominus 1) - \{9, 16\}\,, & (G_{51} \ominus 1) - \{8, 9\} &
\end{array}
$$

  are all bipartite. Thus, $r_+(G_i) \leq 3$ for every $i \in \{2, 3, 28, 29, 31, 32, 33, 45, 47, 50, 51\}$.
- The graphs

$$
\begin{array}{lll}
(G_{14} \ominus 1) - \{10, 13, 14\}\,, & (G_{18} \ominus 1) - \{6, 9, 10\}\,, & (G_{36} \ominus 1) - \{7, 8, 11\}\,, \\
(G_{37} \ominus 1) - \{7, 12, 13\}\,, & (G_{38} \ominus 1) - \{8, 9, 11\}\,, & (G_{39} \ominus 1) - \{8, 10, 12\}\,, \\
(G_{40} \ominus 1) - \{7, 8, 10\} & &
\end{array}
$$

  are all bipartite. Thus, $r_+(G_i) \leq 4$ for every $i \in \{14, 18, 36, 37, 38, 39, 40\}$.
- For every $i \in \{35, 41, 53, 54, 57, 58\}$, observe that $G_i \ominus 1$ has exactly 12 vertices, all of which have degree at least 3. This implies that $G_i \ominus 1$ is not 4-minimal, as it follows from Theorem 4 that every 4-minimal graph has at least one vertex with degree 2. Hence $r_+(G_i \ominus 1) \leq 3$, which implies that $r_+(G_i) \leq 4$.

Thus, there does not exist a vertex-transitive graph on at most 19 vertices with LS_+-rank 5, which implies that $\overline{n}_+(5) \geq 20$. □

We remark that there are 267 connected vertex-transitive graphs which satisfy the degree bounds from Lemma 33 with $n = 20$ and $\ell = 5$. While the observations we used in the proof of Proposition 39 can be used to show that the majority of them have LS_+-rank at most 4, checking every graph in this set for a possible graph with LS_+-rank 5 is still a non-trivial task.

6. Concluding remarks and future work

In this manuscript, we took a deep dive into ℓ-minimal graphs, discovering many new instances in the cases of $\ell = 3$ and $\ell = 4$. In particular, we are somewhat confident that we have found every 3-minimal graph (Conjecture 27). We also identified aspects of our findings which align with our pre-existing understanding of ℓ-minimal graphs, such as the prevalence of the stretched cliques and the relevance of the clique number of a graph. The continuation of patterns found in the newly-discovered ℓ-minimal graphs led to Conjectures 30, 31, and 32. On the other hand, we also saw many surprises, such as the discovery of many ℓ-minimal graphs which are not stretched cliques, the most striking of which perhaps being a 4-minimal graph which does not contain K_6 as a graph minor ($G_{14,12}$ from Figure 14). We hope that our findings will lead to fresh insights in the study of lift-and-project relaxations of the stable set polytope of graphs, as well as in closely-related problems. In addition, we also believe the framework of numerical certificates developed in this manuscript can be adapted for very reliably and rigorously verifying solutions in many other convex optimization problems.

To conclude, we mention several open problems which are relevant to our work herein.

Problem 40. Given $\ell \in \mathbb{N}$, what are the maximum and minimum possible edge densities of an ℓ-minimal graph?

Given $\ell \in \mathbb{N}$, define

$$d^-(\ell) := \min \left\{ \frac{|E(G)|}{\binom{|V(G)|}{2}} : G \text{ is an } \ell\text{-minimal graph} \right\},$$

$$d^+(\ell) := \max \left\{ \frac{|E(G)|}{\binom{|V(G)|}{2}} : G \text{ is an } \ell\text{-minimal graph} \right\}.$$

We first raised the problem of computing $d^-(\ell)$ and $d^+(\ell)$ in [AT24a], and have made some progress on this front since with Theorem 7 and the ℓ-minimal graphs discovered herein. Here is what we currently know about these two quantities for $\ell \leq 4$:

ℓ	1	2	3	4
$d^-(\ell)$	$\frac{3}{3}$	$\frac{8}{15}$	$\frac{14}{36}$	$\leq \frac{21}{66}$
$d^+(\ell)$	$\frac{3}{3}$	$\frac{9}{15}$	$\geq \frac{19}{36}$	$\geq \frac{29}{66}$

In particular, the new 3- and 4-minimal graphs discovered in this manuscript seem to suggest that $d^-(\ell)$ is attained by a sparse stretched clique for every $\ell \geq 1$, while $d^+(\ell)$ is likely not attained by a stretched clique for $\ell \geq 3$.

Problem 41. Can we characterize exactly when the vertex-stretching operation is LS_+-rank increasing?

Recall that the vertex-stretching operation is generally LS_+-rank non-decreasing (Theorem 3). With Theorem 7, Proposition 19, and the new ℓ-minimal graphs discovered herein, we now know of a range of situations where the vertex-stretching operation increases the LS_+-rank of the underlying graph by one. It would be interesting to characterize exactly when the operation is LS_+-rank increasing, as well as find out whether it is possible for a vertex-stretching operation to increase the rank of a graph by two or more.

Problem 42. Can we develop better tools for proving rank upper bounds?

In this manuscript, we proposed the framework of LS_+^ℓ certificate packages to help establish LS_+-rank lower bounds. However, we currently do not know of an analogous tool for proving rank upper bounds. For instance, it would be helpful to develop theoretical and/or computational tools to show that the graphs we saw with $r_+(G) \lesssim \ell$ indeed satisfy $r_+(G) \leq \ell$. Progress in this direction will solidify our understanding of ℓ-minimal graphs, and represents a step towards a combinatorial characterization of these graphs.

On a related topic, we can adopt the notions of LS_+^ℓ certificate packages to the duals of the SDPs we have considered in this work. Doing so has the potential of generating more reliable certificates to conclude that $\gamma_\ell(G, a) = 1$.

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